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M-Theory Dynamics On A Manifold of G_2 Holonomy

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We analyze the dynamics of M-theory on a manifold of G_2 holonomy that is developing a conical singularity. The known cases involve a cone on $\mathbf{CP}^3$, where we argue that the dynamics involves restoration of a global symmetry, $SU(3)/U(1)^2$, where we argue that there are phase transitions among three possible branches corresponding to three classical spacetimes, and $\mathbf{S}^3 \times \mathbf{S}^3$ and its quotients, where we recover and extend previous results about smooth continuations between different spacetimes and relations to four-dimensional gauge theory.

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1. Introduction

In studying supersymmetric compactifications of string theory, one of the important issues is the behavior at a singularity. For example, in compactifications of M-theory and of Type II superstring theory, an important role is played by the $A-D-E$ singularities of a K3 surface and by various singularities of a Calabi-Yau threefold. Singularities of heterotic string or D-brane gauge fields are also important, though in the present paper we focus on metric singularities.

The basic questions about string theory and M-theory dynamics at a classical singularity are familiar. What happens in the quantum theory when a classical singularity develops? Does extra gauge symmetry appear? Are there new massless particles? Does the quantum theory flow to a nontrivial infrared fixed point? Is it possible to make a transition to a different classical spacetime by following the behavior of the quantum theory through a classical singularity? If such transitions are possible, do they occur smoothly (as in the case of the classical "flop" of Type II superstring theory [1,2]) or via a phase transition to a different branch of the moduli space of vacua (as in the case of the Type II conifold transition [3,4])?

Usually, the essential phenomena occurring at a singularity are local in nature, independent of the details of a global spacetime in which the singularity is embedded. The most basic singularities from which more elaborate examples are built are generally *conical* in nature. In n dimensions, a conical metric takes the general form

$$ds^2 = dr^2 + r^2 d\Omega^2, \tag{1.1}$$

where r is the "radial" coordinate, and $d\Omega^2$ is a metric on some compact $(n-1)$-manifold Y. An n-manifold X with such a metric is said to be a cone on Y; X has a singularity at the origin unless $Y = \mathbf{S}^{n-1}$ and $d\Omega^2$ is the standard round metric. For example, the $A-D-E$ singularity in real dimension four is a cone on $\mathbf{S}^3/\Gamma$, with $\mathbf{S}^3$ a three-sphere and Γ a finite subgroup of $SU(2)$.

In this example, as in many others, the $A-D-E$ singularity can be resolved (or deformed) to make a smooth four-manifold $\widehat{X}$ that is only asymptotically conical. $\widehat{X}$ carries a hyper-Kahler metric that depends on a number of parameters or moduli. $\widehat{X}$ is smooth for generic values of the moduli, but becomes singular when one varies the moduli so that $\widehat{X}$ is exactly, not just asymptotically, conical.

The present paper is devoted to analyzing M-theory dynamics on a seven-manifold of G_2 holonomy that develops an isolated conical singularity. The motivation for studying G_2-manifolds is of course that G_2 holonomy is the condition for unbroken supersymmetry in four dimensions. This is also the reason that it is possible to get interesting results about this case.

There are probably many possibilities for an isolated conical singularity of a G_2-manifold, but apparently only three simply-connected cases are known [5,6]: a cone on $\mathbf{CP}^3$, $SU(3)/U(1) \times U(1)$, or $\mathbf{S}^3 \times \mathbf{S}^3$ can carry a metric of G_2 holonomy; each of these cones can be deformed to make a smooth, complete, and asymptotically conical manifold X of G_2 holonomy. We will study the behavior of M-theory on these manifolds, as well as on additional examples obtained by dividing by a finite group.

The case of a cone on $\mathbf{S}^3 \times \mathbf{S}^3$ or a quotient thereof has been studied previously [7-10] and found to be rather interesting. This example, in fact, is related to earlier investigations of dualities involving fluxes and branes in topological [11] and ordinary [12-14] strings. We will reexamine it in more detail, and also investigate the other examples, which turn out to be easier to understand.

In section two, we introduce the examples, describe some of their basic properties, and make a proposal for the dynamics of the first two examples. According to our proposal, the dynamics involves in one case the restoration of a global symmetry in the strong coupling region, and in the second case a phase transition between three different branches that represent three different classical spacetimes. In section three, we give evidence for this proposal by relating the manifolds of G_2 holonomy to certain configurations of branes in $\mathbf{C}^3$ that have been studied as examples of singularities of special Lagrangian threefolds [15]. (For somewhat analogous quotients, see [16] and [17].) By a slight extension of these arguments, we also give simple examples of four-dimensional chiral fermions arising from models of G_2 holonomy.

In section four, we analyze the more challenging example, involving the cone on $\mathbf{S}^3 \times \mathbf{S}^3$. In this example, refining the reasoning in [8], we argue that there is a moduli space of theories of complex dimension one that interpolates smoothly, without a phase transition, between three different classical spacetimes. To describe the interpolation precisely, we introduce some natural physical observables associated with the deviation of the geometry at infinity from being precisely conical. Using these observables together with familiar ideas of applying holomorphy to supersymmetric dynamics [18], we give a precise description of the moduli space.

In section five, we compare details of the solution found in section four to topological subtleties in the membrane effective action.

In section six, following [7,8], we consider further examples obtained by dividing by a finite group Γ. In one limit of the models considered, there is an effective four-dimensional gauge theory with a gauge group of type A, D, or E. We again give a precise description of the moduli space in these examples, in terms of the natural observables. For the A series, this enables us to put on a more precise basis some observations made in [8] about the relation of the classical geometry to chiral symmetry breaking. For the D and E series, there is a further surprise: the model interpolates between different possible classical gauge groups, for example $SO(8+2n)$ and $Sp(n)$ in the case of the D series, with generalizations of this statement for the E series. In the case of the E series, the analysis makes contact with recent developments involving commuting triples and M-theory singularities [19].

Our discussion will be relevant to M-theory on a compact manifold of G_2 holonomy if (as is likely but not yet known) such a compact manifold can develop a conical singularity of the types we consider. The known techniques of construction of compact manifolds of G_2 holonomy are explained in detail in a recent book [20].

The asymptotically conical manifolds of G_2 holonomy that we study have been re-examined in recent papers [21,22], along with asymptotically conical metrics with other reduced holonomy groups [23]. Some features explored there will be relevant below, and other aspects are likely to be important in generalizations. For additional work on M-theory on manifolds of G_2 holonomy, see [25-42].

2. Known Examples And Their Basic Properties

In studying M-theory on a manifold X that is asymptotic to a cone on Y, the problem is defined by specifying the fields at infinity, and in particular by specifying Y. The fields are then free to fluctuate in the interior. Here the phrase "in the interior" means that one should look at those fluctuations that decay fast enough at infinity to have finite kinetic energy. Variations of the fields that would have infinite kinetic energy are nondynamical; their values are specified at infinity as part of the definition of the problem. They are analogous to coupling constants in ordinary four-dimensional field theory. The problem of dynamics is to understand the behavior of the fluctuations. Quantum mechanically, one type of fluctuation is that there might be different possible X's once Y is given.

A similar dichotomy holds for symmetries. The symmetries of the problem are the symmetries of the fields at infinity, that is, on Y. An unbroken symmetry is a symmetry that leaves fixed the fields in the interior.

Much of our work in the present section will be devoted to identifying the fluctuating fields and the couplings, the symmetries and the unbroken symmetries, for the various known examples. In describing the examples, we largely follow the notation of [6].

2.1. $\mathbf{R}^3$ *Bundles Over Four-Manifolds*

The starting point for the first two examples is a four-manifold M with self-dual Weyl curvature and a positive curvature Einstein metric, normalized so that $R_{\alpha\beta} = 3g_{\alpha\beta}$. We write the line element of M as $h_{\alpha\beta}dx^\alpha dx^\beta$, where h is the metric and x^α are local coordinates. In practice, the known possibilities for M are $\mathbf{S}^4$ and $\mathbf{CP}^2$. (In the case of $\mathbf{CP}^2$, the orientation is taken so that the Kahler form of $\mathbf{CP}^2$ is considered self-dual.) At a point in M, the space of anti-self-dual two-forms is three-dimensional. The bundle X of anti-self-dual two-forms is accordingly a rank three real vector bundle over M; it carries an $SO(3)$ connection A that is simply the positive chirality part of the spin connection of M written in the spin one representation.

X admits a complete metric of G_2 holonomy:

$$ds^2 = \frac{dr^2}{1-(r_0/r)^4} + \frac{r^2}{4}(1-(r_0/r)^4)|d_Au|^2 + \frac{r^2}{2}\sum_{\alpha,\beta=1}^4 h_{\alpha\beta}dx^\alpha dx^\beta. \tag{2.1}$$

Here u^i are fiber coordinates for the bundle $X \to M$, and d_Au is the covariant derivative $d_Au_i = du_i + \epsilon_{ijk}A_ju_k$. Also, r_0 is an arbitrary positive parameter with dimensions of length, and r is a "radial" coordinate with $r_0 \leq r \leq \infty$.

This metric is asymptotic to a cone on a six-manifold Y that is a two-sphere bundle over M. Indeed, for $r \to \infty$, we can drop the r_0/r terms, and then the metric takes the general form $dr^2 + r^2d\Omega^2$, where here $d\Omega^2$ is a metric on Y. Y is the subspace of X with $\sum_i u_i^2 = 1$ and is known as the twistor space of M.

In fact, we can be more specific: the metric on X differs from a conical metric by terms of order $(r_0/r)^4$, for $r \to \infty$. The exponent 4 is greater than half of the dimension of X, and this has the following important consequence. Let g be the metric of X, and δg the variation of g with respect to a change in r_0. Define the $\mathbf{L}^2$ norm of δg by

$$|\delta g|^2 = \int_X d^7x\sqrt{g}\, g^{ii'}g^{jj'}\delta g_{ij}\delta g_{i'j'}. \tag{2.2}$$

Since $\delta g/g \sim r^{-4}$, and we are in seven dimensions, we have

$$|\delta g|^2 < \infty. \tag{2.3}$$

This means that in M-theory on $\mathbf{R}^4 \times X$, the kinetic energy associated with a fluctuation in r_0 is finite; this fluctuation gives rise to a massless scalar field a in four dimensions.

Supersymmetrically, this massless scalar field must be completed to a massless chiral multiplet Φ. Apart from zero modes of the metric, massless scalars in four dimensions can arise as zero modes of the three-form field C of eleven-dimensional supergravity. In the present example, an additional massless scalar arises because there is on X an $\mathbf{L}^2$ harmonic three-form ω, constructed in [22]. This additional scalar combines with a to make a complex scalar field that is the bosonic part of Φ. The superpotential of Φ vanishes identically for a reason that will be explained later, so the expectation value of Φ parameterizes a family of supersymmetric vacua.

The topological interpretation of ω will be of some interest. ω is zero as an element of $H^3(X;\mathbf{R})$, since in fact (X being contractible to $\mathbf{S}^4$ or $\mathbf{CP}^2$) that group vanishes. The key feature of ω topologically is that

$$\int_{\mathbf{R}^3} \omega \neq 0, \tag{2.4}$$

where the integral is taken over any fiber of the fibration $Y \to M$. Since $\mathbf{R}^3$ is noncompact, this means that ω can be associated with an element of the compactly supported cohomology $H^3_{cpct}(X;\mathbf{R})$. The physical meaning of this will become clear presently.

Geometrical Symmetries

Now we want to discuss the symmetries of these models. Some symmetries arise from geometrical symmetries of the manifold Y. They will be examined in detail when we consider specific examples.

For now, we merely make some general observations about geometrical symmetries. A symmetry of X may either preserve its orientation or reverse it. When X is asymptotic to a cone on Y, a symmetry reverses the orientation of X if and only if it reverses the orientation of Y. In M-theory on $\mathbf{R}^4 \times X$, a symmetry of X that reverses its orientation is observed in the effective four-dimensional physics as an "R-symmetry" that changes the sign of the superpotential. [1] Orientation-preserving symmetries of X are not R-symmetries; they leave the superpotential invariant.

For example, the symmetry $\tau : u \to -u$ of (2.1) reverses the orientation of X, so it is an R-symmetry. Since it preserves the modulus $|\Phi|$, it transforms $\Phi \to e^{i\alpha}\Phi$ for some constant α. We can take α to be zero, because as we will presently see, these models also have a $U(1)$ symmetry (coming from gauge transformations of the C-field) that rotates the phase of Φ without acting as an R-symmetry. Existence of a symmetry that leaves Φ fixed and reverses the sign of the superpotential means that the superpotential is zero for a model of this type. This is consistent with the fact that, as the third homology of X vanishes in these examples, there are no membrane instantons that would generate a superpotential.

We should also consider geometrical symmetries of the first factor of $\mathbf{R}^4 \times X$. Apart from the connected part of the Poincaré group, we must consider a "parity" symmetry, reflecting one of the $\mathbf{R}^4$ directions in $\mathbf{R}^4 \times X$. This exchanges chiral and anti-chiral superfields, changes the sign of the C-field, and maps the chiral superfield Φ to its complex conjugate $\overline{\Phi}$. By contrast, symmetries of $\mathbf{R}^4 \times X$ that act only on X, preserving the orientation of $\mathbf{R}^4$, give holomorphic mappings of chiral superfields to chiral superfields.

Symmetries From C-Field

Important additional symmetries arise from the M-theory three-form C. As always in gauge theories, global symmetries come from gauge symmetries whose generators do not vanish at infinity but that leave the fields fixed at infinity. (Gauge symmetries whose generators vanish at infinity can be neglected as they act trivially on all physical excitations. Gauge transformations that do not leave the fields fixed at infinity are not really symmetries of the physics.)

For the M-theory three-form C, the basic gauge transformation law is $\delta C = d\Lambda$, where Λ is a two-form. So a symmetry generator obeys $d\Lambda = 0$ at infinity. In M-theory on $\mathbf{R}^4 \times X$, with X being asymptotic to a cone on a six-manifold Y, the global symmetry group coming from the C-field is therefore $K = H^2(Y; U(1))$. What subgroup of K leaves

[1] For example, a symmetry of X of order two that preserves the G_2 structure and reverses the orientation of X squares to -1 on spinors, so its eigenvalues are $\pm i$. In M-theory, the gravitino field on $\mathbf{R}^4 \times X$ is real, so such a symmetry acts as $\pm i$ on positive chirality gravitinos on $\mathbf{R}^4$ and as $\mp i$ on their complex conjugates, the negative chirality gravitinos. Since it transforms the two chiralities oppositely, it is an R-symmetry.

the vacuum invariant? A two-form Λ generates an unbroken symmetry if, in a gauge transformation generated by Λ, $0 = \delta C = d\Lambda$ everywhere, not just at infinity. So an element of G is unbroken if it is obtained by restricting to Y an element of $H^2(X;U(1))$. Such elements of K form the subgroup L of unbroken symmetries.

In practice, for X an $\mathbf{R}^3$ bundle over a four-manifold M, a spontaneously broken symmetry arises for Λ such that

$$\int_{\mathbf{S}^2} \Lambda \neq 0, \tag{2.5}$$

where here $\mathbf{S}^2$ is the "sphere at infinity" in one of the $\mathbf{R}^3$ fibers. Indeed, when this integral is nonzero, Λ cannot be extended over X as a closed form, since $\mathbf{S}^2$ is a boundary in X – it is the boundary of a fiber. But Λ can be extended over X somehow, for example, by multiplying it by a function that is 1 at infinity on X and zero in the "interior." After picking such an extension of Λ, we transform C by $\delta C = d\Lambda$, and we have $\int_{\mathbf{R}^3} \delta C = \int_{\mathbf{S}^2} \Lambda$, so

$$\int_{\mathbf{R}^3} \delta C \neq 0. \tag{2.6}$$

Under favorable conditions, and in particular in the examples considered here, we can pick Λ so that δC is harmonic. Then the massless scalar in four dimensions associated with the harmonic three-form δC is a Goldstone boson for the symmetry generated by Λ. That in turn gives us the physical meaning of (2.4): the massless scalar in four dimensions associated with the harmonic three-form ω is the Goldstone boson of a spontaneously broken symmetry whose generator obeys (2.5).

2.2. First Example And Proposal For Dynamics

There are two known examples of this type. In the first example, $M = \mathbf{S}^4$. The three-plane bundle X over M is asymptotic to a cone on $Y = \mathbf{CP}^3$. This arises as follows.

$\mathbf{CP}^3$ admits two different homogeneous Einstein metrics. The usual Fubini-Study metric has $SU(4)$ symmetry, and the second one, which is relevant here, is invariant under the subgroup $Sp(2)$ of $SU(4)$.[2] Thus, $\mathbf{CP}^3$ can be viewed as the homogeneous space $SU(4)/U(3)$, but for the present purposes, it can be more usefully viewed as the homogeneous space $Sp(2)/SU(2)\times U(1)$, where $SU(2)\times U(1) \subset SU(2)\times SU(2) = Sp(1)\times Sp(1) \subset Sp(2)$.

[2] Our notation is such that $Sp(1) = SU(2)$. The preceding statements about the symmetry groups can be refined slightly; the groups that act faithfully on $\mathbf{CP}^3$ are, respectively, $SU(4)/\mathbf{Z}_4$ and $Sp(2)/\mathbf{Z}_2$.

If we divide $Sp(2)$ by $SU(2) \times SU(2)$, we are imposing a stronger equivalence relation than if we divide by $SU(2) \times U(1)$. So defining a six-manifold and a four-manifold Y and M by

$$Y = Sp(2)/U(1) \times SU(2), \quad M = Sp(2)/SU(2) \times SU(2), \tag{2.7}$$

Y fibers over M with fibers that are copies of $SU(2)/U(1) = \mathbf{S}^2$. In fact, as $SO(5) = Sp(2)/\mathbf{Z}_2$ and $SO(4) = SU(2) \times SU(2)/\mathbf{Z}_2$, M is the same as $SO(5)/SO(4) = \mathbf{S}^4$. If we replace the $\mathbf{S}^2$ fibers of $Y \to \mathbf{S}^4$ by $\mathbf{R}^3$'s, we get an asymptotically conical seven-manifold X.

X is the bundle of anti-self-dual two-forms over $\mathbf{S}^4$. Indeed, an anti-self-dual two-form at a point on $\mathbf{S}^4$ breaks the isotropy group of that point from $SU(2) \times SU(2)$ to $SU(2) \times U(1)$. If we restrict to unit anti-self-dual two-forms, we get the six-manifold Y. $M = \mathbf{S}^4$ admits the standard "round" Einstein metric, and an asymptotically conical metric of G_2 holonomy on X is given in (2.1).

Geometrical Symmetries

Now let us work out the symmetries of Y, and of X. Geometrical symmetries of Y can be interpreted rather like the symmetries of the C-field that were discussed in section 2.1. Symmetries of Y are symmetries of M-theory on $\mathbf{R}^4 \times X$. A symmetry of Y that extends over X is an unbroken symmetry. A symmetry of Y that does not extend over X is spontaneously broken; it maps one X to another possible X. In the particular example at hand, we will see that all of the symmetries of Y extend over X, but that will not be so in our other examples.

We can represent Y as the space of all $g \in Sp(2)$, with the equivalence relation $g \cong gh$ for $h \in SU(2) \times U(1)$. In this description, it is clear that Y is invariant under the left action of $Sp(2)$ on g (as noted in a footnote above, it is the quotient $Sp(2)/\mathbf{Z}_2 = SO(5)$ that acts faithfully). These symmetries also act on M and X. Additional symmetries of Y come from right action by elements $w \in Sp(2)$ that "centralize" $H = SU(2) \times U(1)$, that is, for any $h \in H$, $w^{-1}hw \in H$. (We should also consider outer automorphisms of $Sp(2)$ that centralize H, but $Sp(2)$ has no outer automorphisms.) Any $w \in H$ centralizes H, but acts trivially on Y. There is only one nontrivial element w of $Sp(2)$ that centralizes H. It is contained in the second factor of $SU(2) \times SU(2) \subset Sp(2)$. If we represent the $U(1)$ factor in $H = SU(2) \times U(1) \subset SU(2) \times SU(2)$ by the diagonal elements

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{2.8}$$

of $SU(2)$, then we can represent w by the $SU(2)$ element

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.9}$$

that anticommutes with the generator of $U(1)$.

w acts trivially on the four-manifold $M = Sp(2)/SU(2)\times SU(2)$, since it is contained in $SU(2)\times SU(2)$. To show that all symmetries of Y extend as symmetries of X, we must, as X is manifestly $Sp(2)$-invariant, find a $\mathbf{Z}_2$ symmetry of X that acts trivially on M. It is simply the R-symmetry τ that we discussed earlier: multiplication by -1 on the $\mathbf{R}^3$ fibers of $X \to M$, or in other words the transformation $u \to -u$.

Dynamics

Now let us try to guess the dynamics of the chiral superfield Φ. Φ is essentially

$$\Phi = V e^{i\int_{\mathbf{R}^3} C}, \tag{2.10}$$

where $V \sim r_0^4$ is the volume of the $\mathbf{S}^4$ at the "center" of X, and the integral in the exponent is taken over a fiber of $X \to M$.

The geometrical symmetry τ of Y extends over X, as we have seen, regardless of r_0. So it acts trivially on the modulus of Φ. It also leaves fixed the argument of Φ. (Concretely, τ reverses the orientation of $\mathbf{R}^3$, but also, since it reverses the orientation of the overall spacetime $\mathbf{R}^4 \times X$, transforms C with an extra factor of -1. The net effect is to leave fixed $\int_{\mathbf{R}^3} C$.) Hence, this symmetry plays no role in the low energy dynamics.

We also have the "parity" symmetry of $\mathbf{R}^4 \times X$, reflecting the first factor. This symmetry acts by $\Phi \to \overline{\Phi}$. (Indeed, such a transformation maps $C \to -C$ with no action on $\mathbf{R}^3$.)

The important symmetries for understanding this problem come from the symmetries of the C-field. In the present example, with $Y = \mathbf{CP}^3$, we have $H^2(Y; U(1)) = U(1)$, so the symmetry group is $K = U(1)$. On the other hand, X is contractible to $M = \mathbf{S}^4$, and $H^2(\mathbf{S}^4; U(1)) = 0$, so K is spontaneously broken to nothing.

Also, since the second homology group of M is trivial, the second homology group of Y is generated by a fiber of the $\mathbf{S}^2$ fibration $Y \to M$. The generator of K is accordingly derived from a two-form Λ with

$$\int_{\mathbf{S}^2} \Lambda \neq 0. \tag{2.11}$$

As explained in section 2.1, a symmetry generated by such a Λ shifts the value of $\int_{\mathbf{R}^3} C$, and hence acts on Φ by

$$\Phi \to e^{i\alpha}\Phi. \tag{2.12}$$

This shows explicitly the spontaneous breaking of K.

Supergravity gives a reliable account of the dynamics for large $|\Phi|$, that, is, for large V. We want to guess, using the symmetries and holomorphy, what happens in the quantum regime of small $|\Phi|$. In the present case, there is a perfectly obvious guess. Since it has an action of the global symmetry $K = U(1)$, acting by "rotations" near infinity, the moduli space must have genus zero. Its only known "end" is $\Phi \to \infty$ and – as ends of the moduli space should be visible semiclassically – it is reasonable to guess that this is the only end. If so, the moduli space must simply be the complex Φ-plane, with $\Phi = 0$ as a point at which the global symmetry K is restored. Our minimal conjecture for the dynamics is that there are no extra massless particles for strong coupling; the dynamics remains infrared-free; and the only qualitative phenomenon that occurs for small volume is that at a certain point in the moduli space of vacua, the global $U(1)$ symmetry is restored.

In section 3, we will give supporting evidence for this proposal by comparing the model to a Type IIA model with D-branes.

2.3. A Second Model

In the second example, $Y = SU(3)/T$, where $T = U(1)\times U(1) = U(1)^2$ is the maximal torus of $SU(3)$. Moreover, $M = SU(3)/U(2) = \mathbf{CP}^2$, and X is the bundle of anti-self-dual two-forms over $\mathbf{CP}^2$.

Again we must be careful in discussing the metric and symmetries of Y. First let us look at the geometrical symmetries. The nontrivial symmetries of interest are outer automorphisms of $SU(3)$ that centralize the maximal torus T, and right action by elements of $SU(3)$ that centralize T. Actually, the centralizer of the maximal torus in any compact connected Lie group is the Weyl group W, which for $SU(3)$ is the group Σ_3 of permutations of three elements. If one thinks of an element of $SU(3)$ as a 3×3 matrix, then elements of W are 3×3 permutation matrices (times ± 1 to make the determinant 1).

$SU(3)$ also has an outer automorphism of complex conjugation, which centralizes T (it acts as -1 on the Lie algebra of T). It commutes with Σ_3 (since the permutation matrices generating Σ_3 are real). This gives a symmetry of Y that extends as a symmetry of X (acting by complex conjugation on $M = \mathbf{CP}^2$) regardless of the value of Φ. Because it is a symmetry for any value of the chiral superfield Φ, it decouples from the low energy dynamics and will not be important. By contrast, the Σ_3 or "triality" symmetry is very important, as we will see.

Y admits a well-known homogeneous Kahler-Einstein metric, but that is not the relevant one for the metric (2.1) of G_2 holonomy on X. The reason for this is as follows. The notion of a Kahler metric on Y depends on a choice of complex structure. The (complexified) tangent space of Y has a basis in one-to-one correspondence with the nonzero roots of $SU(3)$. Picking a complex structure on Y is equivalent to picking a set of positive roots. The Weyl group permutes the possible choices of what we mean by positive roots, so it is not natural to expect a Kahler metric on Y to be Σ_3-invariant. In fact, the subgroup of Σ_3 that leaves fixed a Kahler metric is $\Sigma_2 = \mathbf{Z}_2$, the group of permutations of *two* elements. Indeed, if we exchange the positive and negative roots of $SU(3)$ (by making a Weyl transformation that is a reflection with respect to the highest root), this will reverse the complex structure of Y; but a metric that is Kahler for one complex structure is also Kahler for the opposite complex structure. Accordingly, the standard Kahler metric of Y is invariant under the subgroup Σ_2 of Σ_3.

Y also admits $SU(3)$-invariant metrics that are invariant under the full Σ_3. In fact, to give an $SU(3)$-invariant metric on Y, we first give a T-invariant metric on the tangent space at a point, and then transport it by $SU(3)$. There is up to a scalar multiple a unique T-invariant metric on the tangent space that is also Σ_3-invariant: it assigns the same length to each nonzero root of $SU(3)$. So an $SU(3) \times \Sigma_3$-invariant metric g_{IJ} on Y is uniquely determined up to a scale. This metric is hermitian but not Kahler (for each of the complex structures of Y).

Uniqueness up to scale of the $SU(3) \times \Sigma_3$-invariant metric implies that this metric is an Einstein metric. Indeed, the Ricci tensor R_{IJ} derived from g_{IJ} is again a symmetric tensor with the same $SU(3) \times \Sigma_3$ symmetry, so it must be a multiple of g_{IJ}. The G_2 metric on X is asymptotic to a cone on Y, where Y is endowed with this Σ_3-invariant Einstein metric.

Now, let us ask what subgroup of Σ_3 is preserved by X. X is an $\mathbf{R}^3$ bundle over $M = \mathbf{CP}^2 = SU(3)/SU(2) \times U(1)$. M in fact has no symmetries that commute with $SU(3)$, since $SU(2) \times U(1)$ has no nontrivial centralizer in $SU(3)$. X has a $\mathbf{Z}_2$ symmetry that commutes with $SU(3)$ and acts trivially on M: it is the transformation $u \to -u$ in (2.1). So picking a particular X that is bounded by Y breaks the Σ_3 symmetry of Y down to $\Sigma_2 = \mathbf{Z}_2$. The $\mathbf{Z}_2$ in question is an antiholomorphic transformation of Y that reverses the complex structure.

In essence, though Y admits various choices of complex structure and an Einstein metric on Y can be defined without making a choice, to fiber Y over $\mathbf{CP}^2$ one must pick (up to sign, that is, up to an overall reversal of the complex structure) a distinguished complex structure on Y. So there are three different choices of X, determined by the choice of complex structure on Y.

Next, let us consider the symmetries that originate from the C-field. Consider the fibration $SU(3) \to Y$ with fibers $T = U(1) \times U(1)$. Because the first and second cohomology groups of $SU(3)$ with $U(1)$ coefficients are zero, the spectral sequence for this fibration gives us

$$H^2(Y; U(1)) = H^1(T; U(1)) \tag{2.13}$$

In fact,

$$H^1(T; U(1)) = T^* = U(1) \times U(1), \tag{2.14}$$

where T^* is the dual torus of T. On the other hand, X is contractible to $M = \mathbf{CP}^2$, and $H^2(\mathbf{CP}^2; U(1)) = U(1)$. So the global symmetry group coming from the C-field is $K = T^* = U(1) \times U(1)$, spontaneously broken to $L = U(1)$.

Here, we need to be more precise, because there are really, as we have seen above, three possible choices for X, and each choice will give a different unbroken $U(1)$. We need to know how the unbroken $U(1)$'s are related. We can be more precise using (2.14). Thus, the maximal torus T is the quotient of $\mathbf{R}^2$ by the root lattice Λ of $SU(3)$, and T^* is the quotient by the weight lattice Λ^*. The unbroken subgroup L is a one-parameter subgroup of T^*; such a subgroup is determined by a choice (up to sign) of a primitive weight $w \in \Lambda^*$, which one can associate with the generator of L.

The three possible X's – call them X_1, X_2, and X_3 – are permuted by an element of order three in Σ_3. The corresponding w's are likewise (if their signs are chosen correctly) permuted by the element of order three. Since an element of the Weyl group of order three rotates the weight lattice of $SU(3)$ by a $2\pi/3$ angle, the weights w_1, w_2, and w_3 are permuted by such a rotation, and hence

$$w_1 + w_2 + w_3 = 0. \tag{2.15}$$

On the i^{th} branch, there is an unbroken subgroup of Σ_3 generated by an element τ_i of order 2. An element of order 2 in Σ_3 acts by a reflection on the weight lattice, so its eigenvalues are $+1$ and -1. We can identify which is which. τ_i (which acts on X_i by $u \to -u$ in the notation of (2.1)) acts trivially on the chiral superfield Φ_i, by arguments that we have already seen. So τ_i acts trivially on the Goldstone boson field, which is the argument of Φ_i. Hence τ_i leaves fixed the broken symmetry on the i^{th} branch, and acts by

$$\tau_i(w_i) = -w_i \tag{2.16}$$

on the generator w_i of the unbroken symmetry.

Proposal For Dynamics

Now we will make a proposal for the dynamics of this model.

The fact that the unbroken symmetries are different for the three classical spacetimes X_i implies that one cannot, in this model, smoothly interpolate without a phase transition between the three different limits. Instead, we claim one can continuously interpolate between the different classical spacetimes by passing through a phase transition.

We need a theory with three branches of vacua. On the i^{th} branch, for $i = 1, 2, 3$, there must be a single massless chiral superfield Φ_i. The three branches are permuted by a Σ_3 symmetry. Moreover, there is a global $U(1)\times U(1)$ symmetry, spontaneously broken on each branch to $U(1)$. Σ_3 acts on $U(1) \times U(1)$ like the Weyl group of $SU(3)$ acting on the maximal torus, and the generators w_i of the unbroken symmetries of the three branches add up to zero.

We will reproduce this via an effective theory that contains all three chiral superfields Φ_i, in such a way that there are three branches of the moduli space of vacua on each of which two of the Φ_i are massive. We take Σ_3 to act by permutation of the Φ_i, and we take $K = U(1) \times U(1)$ to act by $\Phi_i \to e^{i\theta_i}\Phi_i$ with $\theta_1 + \theta_2 + \theta_3 = 0$. The minimal nonzero superpotential invariant under $K \times \Sigma_3$ is

$$W(\Phi_1, \Phi_2, \Phi_3) = \lambda\Phi_1\Phi_2\Phi_3 \tag{2.17}$$

with λ a constant. In the classical approximation, a vacuum is just a critical point of W. The equations for a critical point are $0 = \Phi_2\Phi_3 = \Phi_3\Phi_1 = \Phi_1\Phi_2$, and there are indeed three branches of vacua permuted by Σ_3. On the i^{th} branch, for $i = 1, 2, 3$, Φ_i is nonzero and the other Φ's are zero and massive. The three branches meet at a singular point at the origin, where $\Phi_1 = \Phi_2 = \Phi_3 = 0$. Thus, one can pass from one branch to another by going through a phase transition at the origin.

On the branch with, say, $\Phi_1 \neq 0$, the unbroken $U(1)$ is $\Phi_1 \to \Phi_1$, $\Phi_2 \to e^{i\theta}\Phi_2$, $\Phi_3 \to e^{-i\theta}\Phi_3$, generated by a diagonal matrix $w_1 = \mathrm{diag}(0,1,-1)$ acting on the Φ_i. The generators w_2 and w_3 of the unbroken symmetries on the other branches are obtained from w_1 by a cyclic permutation of the eigenvalues, and indeed obey $w_1 + w_2 + w_3 = 0$. Moreover, the nontrivial element of Σ_3 that leaves fixed, say, the first branch is the element τ_1 that exchanges Φ_2 and Φ_3; it maps w_1 to $-w_1$, as expected according to (2.16).

We will give additional evidence for this proposal in section 3, by comparing to a construction involving Type IIA D-branes. For now, we conclude with a further observation about the low energy physics. The theory with superpotential W has an additional $U(1)$ R-symmetry that rotates all of the Φ_i by a common phase. It does not correspond to any exact symmetry of M-theory even at the critical point. In general, when X develops a conical singularity, M-theory on $\mathbf{R}^4 \times X$ might develop exact symmetries that act only on degrees of freedom that are supported near the singularity, but they cannot be R-symmetries. The reason for this last statement is that R-symmetries act nontrivially on the gravitino and the gravitino can propagate to infinity on X.

We could remove the R-symmetry by adding nonminimal terms to the theory, for example an additional term $(\Phi_1\Phi_2\Phi_3)^2$ in W. But such terms are irrelevant in the infrared. So a consequence of our proposal for the dynamics is that the infrared limit of M-theory at the phase transition point has a $U(1)$ R-symmetry that is not an exact M-theory symmetry.

2.4. $\mathbf{R}^4$ *Bundle Over* $\mathbf{S}^3$

Now we move on to the last, and, as it turns out, most subtle example. This is the case of a manifold of G_2 holonomy that is asymptotic at infinity to a cone over $Y = \mathbf{S}^3 \times \mathbf{S}^3$. In this case, the seven-manifold X is topologically $\mathbf{R}^4 \times \mathbf{S}^3$; the "vanishing cycle" that collapses when X develops a conical singularity is the three-manifold $Q = \mathbf{S}^3$ rather than the four-manifold $M = \mathbf{S}^4$ or $\mathbf{CP}^2$ of the previous examples. That is in fact the basic reason for the difference of this example from the previous ones.

All the manifolds in sight – X, Y, and Q – have vanishing second cohomology. So there will be no symmetries coming from the C-field. However, the geometrical symmetries of Y will play an important role.

$Y = \mathbf{S}^3 \times \mathbf{S}^3$ admits an obvious Einstein metric that is the product of the standard round metric on each $\mathbf{S}^3$. Identifying $\mathbf{S}^3 = SU(2)$, this metric has $SU(2)^4$ symmetry, acting by separate left and right action of $SU(2)$ on each factor of $Y = SU(2) \times SU(2)$.

However, the $SU(2)^4$-invariant Einstein metric on Y is not the one that arises at infinity in the manifold of G_2 holonomy.

A second Einstein metric on $\mathbf{S}^3 \times \mathbf{S}^3$ can be constructed as follows. Let a, b, and c be *three* elements of $SU(2)$, constrained to obey

$$abc = 1. \tag{2.18}$$

The constraint is obviously compatible with the action of $SU(2)^3$ by $a \to uav^{-1}$, $b \to vbw^{-1}$, $c \to wcu^{-1}$, with $u, v, w \in SU(2)$. Moreover, it is compatible with a cyclic permutation of a, b, c:

$$\beta : (a, b, c) \to (b, c, a) \tag{2.19}$$

and with a "flip"

$$\alpha : (a, b, c) \to (c^{-1}, b^{-1}, a^{-1}). \tag{2.20}$$

α and β obey $\alpha^2 = \beta^3 = 1$, $\alpha\beta\alpha = \beta^{-1}$. Together they generate the same "triality" group Σ_3 that we met in the previous example.

Another way to see the action of $SU(2)^3 \times \Sigma_3$ on $\mathbf{S}^3 \times \mathbf{S}^3$ is as follows. Consider triples $(g_1, g_2, g_3) \in SU(2)^3$ with an equivalence relation $(g_1, g_2, g_3) \cong (g_1 h, g_2 h, g_3 h)$ for $h \in SU(2)$. The space of equivalence classes is $\mathbf{S}^3 \times \mathbf{S}^3$ (since we can pick h in a unique fashion to map g_3 to 1). On the space Y of equivalence classes, there is an obvious action of Σ_3 (permuting the three g's), and of $SU(2)^3$ (acting on the g's on the left). The relation between the two descriptions is to set $a = g_2 g_3^{-1}$, $b = g_3 g_1^{-1}$, $c = g_1 g_2^{-1}$. The description with the g_i amounts to viewing $\mathbf{S}^3 \times \mathbf{S}^3$ as a homogeneous space G/H, with $G = SU(2)^3$, and $H = SU(2)$ the diagonal subgroup of the product of three $SU(2)$'s. There are no nontrivial elements of G (not in H) that centralize H, and the Σ_3 symmetry group comes from outer automorphisms of G.

On $\mathbf{S}^3 \times \mathbf{S}^3$ there is, up to scaling, a unique metric with $SU(2)^3 \times \Sigma_3$ symmetry, namely

$$d\Omega^2 = -\mathrm{Tr}\left((a^{-1}da)^2 + (b^{-1}db)^2 + (c^{-1}dc)^2\right), \tag{2.21}$$

where Tr is the trace in the two-dimensional representation of $SU(2)$. Just as in our discussion of Einstein metrics on $SU(3)/U(1) \times U(1)$, the uniqueness implies that it is an Einstein metric. We will abbreviate $-\mathrm{Tr}(a^{-1}da)^2$ as da^2, and so write the metric as

$$d\Omega^2 = da^2 + db^2 + dc^2. \tag{2.22}$$

The manifold $X = \mathbf{R}^4 \times \mathbf{S}^3$ admits a complete metric of G_2 holonomy described in [5,6]. It can be written

$$ds^2 = \frac{dr^2}{1-(r_0/r)^3} + \frac{r^2}{72}\left(1-(r_0/r)^3\right)\left(2\,da^2 - db^2 + 2\,dc^2\right) + \frac{r^2}{24}db^2, \tag{2.23}$$

where r is a "radial" coordinate with $r_0 \leq r < \infty$, and r_0 is the "modulus" of the solution.[3] The topology is $\mathbf{R}^4 \times \mathbf{S}^3$ since one of the two $\mathbf{S}^3$'s collapses to zero radius for $r \to r_0$. Near infinity, this metric is asymptotic to that of a cone on $Y = \mathbf{S}^3 \times \mathbf{S}^3$, with the metric on Y being the Einstein metric $d\Omega^2$ described above.

The deviation of the metric on X from a conical form is of order $(r_0/r)^3$ for $r \to \infty$, in contrast to the $(r_0/r)^4$ in the previous examples. The difference is essentially because the vanishing cycle is a three-cycle, while in the previous examples it was a four-cycle. Because a function that behaves as $(r_0/r)^3$ for large r is not square integrable in seven dimensions (on an asymptotically conical manifold), the physical interpretation of the modulus r_0 is very different from what it was in the previous examples. r_0 is not free to fluctuate; the kinetic energy in its fluctuation would be divergent. It should be interpreted as a boundary condition that is fixed at infinity. In the low energy four-dimensional physics, r_0 is a coupling constant. The fact that a fluctuation in r_0 is not square-integrable is related by supersymmetry to the fact that the $SU(2)^3$-invariant harmonic three-form ω on this manifold is not square-integrable [22].

The parameter related to r_0 by supersymmetry is $\theta = \int_Q C$, where $Q = \mathbf{S}^3$ is the "vanishing cycle" at the center of X. (In fact, the harmonic three-form ω obeys $\int_Q \omega \neq 0$, so adding to C a multiple of the zero mode ω shifts θ.) r_0 and θ combine into a complex parameter. In the examples based on $\mathbf{R}^3$ bundles over a four-manifold, there was a massless chiral superfield whose expectation value parameterized a one complex parameter family of vacua. In the present example, there is instead a complex coupling parameter, and for each value of the coupling, there is a unique vacuum. The last statement holds for sufficiently large r_0 because supergravity is valid and gives a unique vacuum with all interactions vanishing in the infrared. Then, by holomorphy, uniqueness of the vacuum should hold for all values of the coupling.

[3] To compare to the notation of [6], let T_i be $SU(2)$ generators with $T_iT_j = \delta_{ij} + i\epsilon_{ijk}T_k$. Then σ_i in eqn. (5.1) of [6] equals $-(i/2)\mathrm{Tr}\,T_i a^{-1}da$. Also, set $\widetilde{b} = b^{-1}$. Then $\Sigma_i = -(i/2)\mathrm{Tr}\,T_i\widetilde{b}^{-1}d\widetilde{b}$. To match (2.23) with the result in [6], one should also solve for c with $c = \widetilde{b}a^{-1}$.

The metric (2.23) is clearly invariant under $\alpha : (a, b, c) \to (c^{-1}, b^{-1}, a^{-1})$, and not under any other nontrivial element of Σ_3. So the choice of X has broken Σ_3 down to the subgroup Σ_2 generated by α. We are thus in a situation that is reminiscent of what we found in section 2.3. There are three different X's, say X_1, X_2, X_3, permuted by the spontaneously broken "triality" symmetry.

To describe the construction of the X_i in topological terms, we reconsider the description of $Y = \mathbf{S}^3 \times \mathbf{S}^3$ in terms of three $SU(2)$ elements g_1, g_2, g_3 with the equivalence relation

$$(g_1, g_2, g_3) = (g_1 h, g_2 h, g_3 h). \tag{2.24}$$

To make a seven-manifold X' bounded by Y, we "fill in" one of the three-spheres. To be more precise, we allow one of the g_i, say g_1, to take values in $\mathbf{B}^4$ – a four-ball bounded by $SU(2)$, to which the right action of $SU(2)$ extends in a natural way – and we impose the same equivalence relation (2.24). If we think of $SU(2)$ as the group of unit quaternions, we can think of $\mathbf{B}^4$ as the space of quaternions of norm no greater than one. Obviously, we could replace g_1 by g_2 or g_3, so we get in this way three different seven-manifolds X_i'. The X_i' are compact seven-manifolds with boundary; if we omit the boundary, we get open seven-manifolds X_i which are the ones that admit asymptotically conical metrics of G_2 holonomy. (Henceforth, we will not generally distinguish X_i and X_i'.) From this construction, it is manifest that X_1, for example, admits a $\mathbf{Z}_2$ symmetry that exchanges g_2 and g_3.

To see the topology of X_1, we just set $h = g_3^{-1}$. So $X_1 = \mathbf{R}^4 \times \mathbf{S}^3$, with $\mathbf{R}^4$ parameterized by g_1 and $\mathbf{S}^3$ by g_2. X_1 is invariant under an element of Σ_3 that exchanges g_2 and g_3. It maps $(g_1, g_2, 1)$ to $(g_1, 1, g_2)$ or equivalently $(g_1 g_2^{-1}, g_2^{-1}, 1)$. Looking at the behavior at the tangent space to a fixed point with $g_2 = 1$, we see that this reverses the orientation of X, and hence is an R-symmetry. So in general, the elements of Σ_3 that are of order two are R-symmetries, just as in the model studied in section 2.3.

Using this topological description of the X_i, we can compare to the Type IIA language that was used in [7,8] and understand from that point of view why there are three X_i. To relate to Type IIA, the idea in [7,8] was to divide by a $U(1)$ subgroup of one of the three $SU(2)$'s, say the first one, and interpret the quotient space as a Type IIA spacetime. If we divide $X_1 = \mathbf{R}^4 \times \mathbf{S}^3$ by a $U(1)$ contained in the first $SU(2)$, it acts only on the $\mathbf{R}^4$ factor, leaving fixed $\{0\} \times \mathbf{S}^3 = \mathbf{S}^3$, where $\{0\}$ is the origin in $\mathbf{R}^4$. The quotient $\mathbf{R}^4/U(1)$ is topologically $\mathbf{R}^3$, where the fixed point at the origin is interpreted in Type IIA as signifying the presence of a D-brane, as in the familiar relation of a smooth M-theory spacetime (the Kaluza-Klein monopole) to a Type IIA $D6$-brane [43]. So $X_1/U(1)$ is an $\mathbf{R}^3$ bundle over $\mathbf{S}^3$ with a brane wrapped on the zero-section; the $\mathbf{R}^3$ bundle over $\mathbf{S}^3$ is the deformation of the conifold. The same $U(1)$, in acting on X_2 or X_3, acts nontrivially (and freely) on the $\mathbf{S}^3$ factor of $\mathbf{R}^4 \times \mathbf{S}^3$, giving a quotient that is an $\mathbf{R}^4$ bundle over $\mathbf{S}^2 = \mathbf{S}^3/U(1)$ (with a unit of RR two-form flux on $\mathbf{S}^2$ because the fibration $\mathbf{S}^3 \to \mathbf{S}^2$ has Euler class one). These quotients of X_2 and X_3 are the two possible small resolutions of the conifold. Thus, the three possibilities – the deformation and the two small resolutions – are familiar in Type IIA. The surprise is, perhaps, the triality symmetry between them in the M-theory description in the case with one sixbrane or unit of flux.

Smooth Continuation?

One of our major conclusions so far is that unlike the models considered in sections 2.2 and 2.3, which have a family of quantum vacua depending on one complex parameter, the present example has a one complex parameter family of possible values of the "coupling constants." For each set of couplings, there is a unique vacuum.

Let $\mathcal{N}$ be a complex Riemann surface parameterized by the possible couplings. Thus, $\mathcal{N}$ might be regarded from a four-dimensional point of view as the moduli space of "theories," while in the other examples, the analogous object would be a moduli space $\mathcal{M}$ of vacua in a fixed theory.

In section 2.3, we argued, in a superficially similar case, that the moduli space $\mathcal{M}$ of vacua has three distinct components $\mathcal{M}_i$, one for each classical spacetime. For the present problem, it has been proposed [8] that the curve $\mathcal{N}$ has only one smooth component, which interpolates between the possible classical limits. We can give a quick argument for this based on the relation to Type IIA that was just explained along with triality symmetry.

In the conformal field theory of the Type IIA conifold, in the absence of RR flux, one can interpolate smoothly between the two small resolutions, without encountering a phase transition [1,2]. On the other hand, the transition to the deformation of the conifold involves a phase transition known as the conifold transition [3,4]. Now what happens if one turns on a unit of RR two-form flux so as to make contact with M-theory on a manifold of G_2 holonomy? The effects of RR flux are proportional to the string coupling constant g_s and so (assuming that the number of flux quanta is fixed as $g_s \to 0$) are negligible in the limit of weak string coupling or conformal field theory. Existence of a smooth interpolation between two limits is a stable statement that is not spoiled by a sufficiently small perturbation, so we can assert that the two small resolutions are smoothly connected also when the RR flux is turned on. What about the deformation? In the presence of precisely one unit of RR flux, we can go to M-theory and use the triality symmetry between the three branches; this at once implies that if two of the branches are smoothly connected to each other, they must be smoothly connected to the third branch.

We will study the curve $\mathcal{N}$ more comprehensively in sections 4 and 6. We will obtain a quantitative description of $\mathcal{N}$, using arguments that also apply to the case of more than one unit of RR flux, where there is no triality symmetry.

2.5. Classical Geometry

We conclude with some more detailed observations on the classical geometry that will be useful in section 4.

We want to describe the three-dimensional homology and cohomology of $Y = \mathbf{S}^3 \times \mathbf{S}^3$ as well as of the X_i. We regard Y as the space $SU(2)^3/SU(2)$, obtained by identifying triples (g_1, g_2, g_3) under right multiplication by $h \in SU(2)$. We let $\widehat{D}_i \subset SU(2)^3$ be the i^{th} copy of $SU(2)$. (We will take the index i to be defined mod 3.) $\widehat{D}_1$, for example, is the set $(g, 1, 1)$, $g \in SU(2)$. In $Y = SU(2)^3/SU(2)$, the $\widehat{D}_i$ project to three-cycles that we will call D_i. As the third Betti number of Y is two, the D_i must obey a linear relation. In view of the triality symmetry of Y, which permutes the D_i, this relation is

$$D_1 + D_2 + D_3 = 0. \tag{2.25}$$

In terms of the description of Y by $SU(2)$ elements a, b, c with $abc = 1$ (where $a = g_2 g_3^{-1}$, $b = g_3 g_1^{-1}$, $c = g_1 g_2^{-1}$), D_1 is $a = 1 = bc$, and the others are obtained by cyclic permutation of a, b, c.

As before, we can embed Y in three different manifolds X_i that are each homeomorphic to $\mathbf{R}^4 \times \mathbf{S}_3$. The third Betti number of X_i is one, so in the homology of X_i, the D_i obey an additional relation. Since X_i is obtained by "filling in" the i^{th} copy of $SU(2)$, the relation is just $D_i = 0$. Thus, the homology of X_i is generated by D_{i-1} or D_{i+1} with $D_{i-1} = -D_{i+1}$. At the "center" of X_i, there is a three-sphere Q_i defined by setting $r = r_0$ in (2.23). In the description of X_i via (g_1, g_2, g_3) (modulo right multiplication by $h \in SU(2)$) where g_i takes values in $\mathbf{R}^4$ and the others in $SU(2)$, Q_i corresponds to setting $g_i = 0$. One can then use right multiplication by h to gauge g_{i+1} or g_{i-1} to 1. Since $g_i = 0$ is homotopic to $g_i = 1$, Q_i is homologous, depending on its orientation, to $\pm D_{i-1}$ and to $\mp D_{i+1}$.

Next, let us look at the intersection numbers of the D_i. Any two distinct D_i intersect only at the point $(g_1, g_2, g_3) = (1, 1, 1)$, so the intersection numbers are ± 1. If we orient Y suitably so that $D_1 \cdot D_2 = +1$, then the remaining signs are clear from triality symmetry:

$$D_i \cdot D_j = \delta_{j,i+1} - \delta_{j,i-1}. \tag{2.26}$$

Note that $D_j \cdot (D_1 + D_2 + D_3) = 0$ for all j, consistent with (2.25).

Finally, let us look at cohomology. The third cohomology group of $SU(2)$ is generated by a three-form $\omega = (1/8\pi^2)\mathrm{Tr}(g^{-1}dg)^3$ that integrates to one. On Y, we consider the forms $e^1 = (1/8\pi^2)\mathrm{Tr}(a^{-1}da)^3$, $e^2 = (1/8\pi^2)\mathrm{Tr}(b^{-1}db)^3$, $e^3 = (1/8\pi^2)\mathrm{Tr}(c^{-1}dc)^3$. (We use the forms $a^{-1}da$, etc., rather than $g_i^{-1}dg_i$, as they make sense on the quotient space $Y = SU(2)^3/SU(2)$.) We have $e^1 = (1/8\pi^2)\left(\mathrm{Tr}(g_2^{-1}dg_2)^3 - \mathrm{Tr}(g_3^{-1}dg_3)^3 + 3d\mathrm{Tr}g_2^{-1}dg_2g_3^{-1}dg_3\right)$, and cyclic permutations of that formula. Integrating the e^i over D_j, we get

$$\int_{D_i} e^j = \delta_{j,i+1} - \delta_{j,i-1}. \tag{2.27}$$

Comparing (2.26) and (2.27), it follows that the map from cohomology to homology given by Poincaré duality is

$$e^j \to D_j. \tag{2.28}$$

In section 6, we will also want some corresponding facts about the classical geometry of $Y_\Gamma = \mathbf{S}^3/\Gamma \times \mathbf{S}^3$. Here Γ is a finite subgroup of $SU(2)$, and we understand Y to be, in more detail, $\Gamma\backslash SU(2)^3/SU(2)$, where as usual $SU(2)$ acts on the right on all three factors of $SU(2)^3$, while Γ acts on the left on only the first $SU(2)$. Thus, concretely, an element of Y_Γ can be represented by a triple (g_1, g_2, g_3) of $SU(2)$ elements, with $g_1 \cong \gamma g_1$ for $\gamma \in \Gamma$ and $(g_1, g_2, g_3) \cong (g_1h, g_2h, g_3h)$ for $h \in SU(2)$. Let us rewrite the relation (2.25) in terms of Y_Γ. The D_i for $i > 1$ project to cycles $D_i' \cong \mathbf{S}^3 \in Y_\Gamma$, but D_1 projects to an N-fold cover of $D_1' = \mathbf{S}^3/\Gamma$, which we can regard as the first factor in $Y_\Gamma = \mathbf{S}^3/\Gamma \times \mathbf{S}^3$. So we have

$$ND_1' + D_2' + D_3' = 0. \tag{2.29}$$

In the manifold $X_{i,\Gamma}$ obtained by "filling in" g_i, there is an additional relation $D_i' = 0$. So the homology of $X_{2,\Gamma}$, for example, is generated by D_1' with $D_2' = 0$ and $D_3' = -ND_1'$. To compute the intersection numbers of the D_i' in Y_Γ, we lift them up to Y and count the intersections there, and then divide by N (since N points on Y project to one on Y_Γ). D_1', D_2', and D_3' lift to D_1, ND_2, ND_3, so the intersections are

$$D_1' \cdot D_2' = -D_1' \cdot D_3' = 1, \quad D_2' \cdot D_3' = N \tag{2.30}$$

(and $D_i' \cdot D_j' = -D_j' \cdot D_i'$ as the cycles are of odd dimension). This is consistent with (2.29).

3. Relation To Singularities Of Special Lagrangian Three-Cycles

3.1. Introduction

In this section we shall investigate in more detail the geometry of our three different six-manifolds Y, namely

$$(\mathrm{I})\ \mathbf{CP}^3,\quad (\mathrm{II})\ \mathrm{SU}(3)/\mathrm{U}(1)^2,\quad (\mathrm{III})\ \mathbf{S}^3\times\mathbf{S}^3, \tag{3.1}$$

and the associated manifolds X of G_2 holonomy.

As we have noted, all of these have Einstein metrics, homogeneous for the appropriate groups, and giving rise to cones with G_2 holonomy. Each of these cones admits a deformation to a smooth seven-manifold X with G_2 holonomy and the same symmetry group.

In general, if one is given a free action of $U(1)$ on a G_2-manifold X, then $X/U(1)$ is a six-manifold with a natural symplectic structure. The symplectic form ω of $X/U(1)$ is obtained by contracting the covariantly constant three-form Υ of X with the Killing vector field K that generates the $U(1)$ action on X. (In other words, $\omega = \pi_*\Upsilon$, where π is the projection $\pi : X \to X/U(1)$.)

In this situation, M-theory on $\mathbf{R}^4\times X$ is equivalent to Type IIA on $\mathbf{R}^4\times X/U(1)$. $X/U(1)$ is only Calabi-Yau if the $U(1)$ orbits on X all have the same length. Otherwise, in Type IIA language, there is a nonconstant dilaton field, and a more general form of the condition for unbroken supersymmetry must be used.

A reduction to Type IIA still exists if the $U(1)$ action on X has fixed points precisely in codimension four. Because of the G_2 holonomy, the $U(1)$ action in the normal directions to the fixed point set always looks like

$$(n_1,n_2)\to(\lambda n_1,\lambda n_2),\ \ \lambda=e^{i\theta}, \tag{3.2}$$

with some local complex coordinates n_1, n_2 on the normal bundle $\mathbf{R}^4\cong\mathbf{C}^2$.[4] The quotient $\mathbf{C}^2/U(1)$, with this type of $U(1)$ action, is isomorphic to $\mathbf{R}^3$, the natural coordinates on $\mathbf{R}^3$ being the hyperkähler moment map $\vec{\mu}$, which in physics notation is written

$$\vec{\mu} = (n, \vec{\sigma} n). \tag{3.3}$$

[4] In the tangent space T_P at a fixed point P, $U(1)$ must act as a subgroup of G_2, so as to preserve the G_2 structure of X. So our statement is that any $U(1)$ subgroup of G_2 that leaves fixed a three-dimensional subspace of T_P acts as in (3.2). Indeed, the Lie algebra of such a $U(1)$ is orthogonal to a nonzero weight in the seven-dimensional representation of G_2, and this uniquely fixes it, up to conjugation.

where $n \in \mathbf{C}^2$, $(\ , \ )$ is a hermitian inner product, and $\vec{\sigma}$ are Pauli matrices, a basis of hermitian traceless 2×2 matrices.

Whenever the fixed points are in codimension four, it follows by using this local model at the fixed points that $X/U(1)$ is a manifold. Moreover, $X/U(1)$ is symplectic; it can be shown using the explicit description in (3.3) and the local form of a G_2 structure that ω is smooth and nondegenerate even at points in $X/U(1)$ that descend from fixed points in X. The fixed point set in $L \subset X$ is three-dimensional (since it is of codimension four) and maps to a three-manifold, which we will also call L, in $X/U(1)$. L is always Lagrangian, but is not always special Lagrangian, just as X is not always Calabi-Yau. Physically, as explained in [43], L is the locus of a D-brane (to be precise, a $D6$-brane whose worldvolume is $\mathbf{R}^4 \times L$).

For any $U(1)$ action on X whose fixed points are in codimension four, this construction gives a way of mapping M-theory on $\mathbf{R}^4 \times X$ to an equivalent Type IIA model. In [8], this situation was investigated in detail for $Y = \mathbf{S}^3 \times \mathbf{S}^3$ and a particular choice of $U(1)$. We shall now investigate a different class of $U(1)$ subgroups that have fixed points of codimension four, as follows:

(I) For $Y = \mathbf{CP}^3$, the connected global symmetry group is $Sp(2)$. We take $U(1) \subset Sp(1) \subset Sp(2)$.

(II) For $Y = SU(3)/U(1)^2$, the connected global symmetry group is $SU(3)$. We take $U(1)$ to consist of elements $\mathrm{diag}(\lambda^{-2}, \lambda, \lambda)$, with $\lambda = e^{i\theta}$.

(III) For $Y = \mathbf{S}^3 \times \mathbf{S}^3$, the connected global symmetry group is $SU(2)^3$, and we take a diagonal $U(1)$ subgroup of the product of the three $SU(2)$'s.

In all three examples, we will show that

$$\begin{aligned} Y/U(1) &\cong \mathbf{S}^5 \\ X/U(1) &\cong \mathbf{R}^6. \end{aligned} \tag{3.4}$$

Moreover, we shall construct explicit smooth identifications in (3.4) which respect the appropriate symmetries.

Since $X/U(1)$ will always be $\mathbf{R}^6$, the interesting dynamics will depend entirely on the fixed point set $L \subset \mathbf{R}^6$. Singularities of X will be mapped to singularities of L. Though we will not get the standard metric on $\mathbf{R}^6$ and neither will L be special Lagrangian, it is reasonable to believe that near the singularities of L, the details of the induced metric on $\mathbf{R}^6$ and the dilaton field are unimportant. If so, then since on a Calabi-Yau manifold, supersymmetry requires that L should be special Lagrangian, it should be (approximately) special Lagrangian near its singularities. Indeed, the singularities of L that we will find are exactly the simplest examples of singularities of special Lagrangian submanifolds of $\mathbf{C}^3$ as investigated in [15], based on earlier work in [44].

The fixed point sets F of the $U(1)$ action on Y will be two-manifolds, which descend to two-manifolds in $Y/U(1) = \mathbf{S}^5$. These will turn out to be

$$\begin{aligned} Y = \mathbf{CP}^3, &\qquad F = \mathbf{S}^2 \cup \mathbf{S}^2 \\ Y = SU(3)/U(1)^2, &\qquad F = \mathbf{S}^2 \cup \mathbf{S}^2 \cup \mathbf{S}^2 \\ Y = \mathbf{S}^3 \times \mathbf{S}^3, &\qquad F = \mathbf{S}^1 \times \mathbf{S}^1. \end{aligned} \tag{3.5}$$

If we take for X simply a cone on Y, then L will be a (one-sided) cone on F. A cone on $\mathbf{S}^2$ is a copy of $\mathbf{R}^3$. So in the first two examples, if X is conical, L consists of two or three copies of $\mathbf{R}^3$, respectively. The $\mathbf{R}^3$'s intersect at the origin, because the $\mathbf{S}^2$'s are linked in $\mathbf{S}^5$.

If we deform to a smooth X, L comes out to be

$$\begin{aligned} Y = \mathbf{CP}^3, &\qquad L = \mathbf{S}^2 \times \mathbf{R} \\ Y = SU(3)/U(1)^2, &\qquad L = \mathbf{S}^2 \times \mathbf{R} \cup \mathbf{R}^3 \\ Y = \mathbf{S}^3 \times \mathbf{S}^3, &\qquad L = \mathbf{S}^1 \times \mathbf{R}^2. \end{aligned} \tag{3.6}$$

Using these facts, we can compare to the claims in section 2 in the following way:

(I) Suppose $Y = \mathbf{CP}^3$ and X is a cone on Y. As explained above, the fixed point set in X corresponds to two copies of $\mathbf{R}^3$, meeting at the origin in $\mathbf{R}^6$. The two $\mathbf{R}^3$'s meet at special angles such that supersymmetry is preserved [45]; a massless chiral multiplet Φ arises at their intersection point from open strings connecting the two $\mathbf{R}^3$'s. Generally, a D-brane supports a $U(1)$ gauge field, but in the present case, because of the noncompactness of the $\mathbf{R}^3$'s, the two $U(1)$'s behave as global symmetries in the effective four-dimensional physics. Φ is neutral under the sum of the two $U(1)$'s, and this sum is irrelevant in the four-dimensional description. So the effective four-dimensional description is given by a chiral multiplet Φ with a single $U(1)$ symmetry. This agrees with what we found in section 2.2. Giving an expectation value to Φ corresponds [46] to deforming the union of the two $\mathbf{R}^3$'s to a smooth, irreducible special Lagrangian manifold of topology $\mathbf{S}^2 \times \mathbf{R}$. This deformation has been described in [44,20] and from a physical point of view in [47]. As stated in (3.6), if we deform the cone on Y to a smooth G_2-manifold X, the resulting fixed point set is indeed $\mathbf{S}^2 \times \mathbf{R}$.

(II) For a cone on $Y = SU(3)/U(1)^2$, the fixed point set is three copies of $\mathbf{R}^3$, meeting at the origin in $\mathbf{R}^6$ in such a way as to preserve supersymmetry. This corresponds to three D-branes whose worldvolumes we call D_i. There are three massless chiral multiplets, say Φ_1, Φ_2, and Φ_3, where Φ_i arises from open strings connecting D_{i-1} and D_{i+1}. Each of the $\mathbf{R}^3$'s generates a $U(1)$ global symmetry, but the sum of the $U(1)$'s decouples, so effectively the global symmetry group of the Φ_i is $U(1)\times U(1)$. A superpotential $\Phi_1\Phi_2\Phi_3$ is generated from worldsheet instantons with the topology of a disc. Because of this superpotential, any one of the Φ_i, but no more, may receive an expectation value. This agrees with the description in section 2.3. On a branch of the moduli space of vacua on which $\langle\Phi_i\rangle \neq 0$, for some i, the union of D_{i-1} and D_{i+1} is deformed just as in case I above to a smooth D-brane with topology $\mathbf{S}^2\times\mathbf{R}$ (and disjoint from D_i), while D_i is undeformed. So $L = \mathbf{S}^2\times\mathbf{R}\cup\mathbf{R}^3$. As stated in (3.6), this is indeed the fixed point set that arises when a cone on Y is deformed to a smooth G_2-manifold.

(III) The cone on $\mathbf{S}^1\times\mathbf{S}^1$ has, as analyzed in [15], three different special Lagrangian deformations, all with topology $\mathbf{S}^1\times\mathbf{R}^2$. (They differ by which one-cycle in $\mathbf{S}^1\times\mathbf{S}^1$ is a boundary in $\mathbf{S}^1\times\mathbf{R}^2$.) These three possibilities for L correspond to the three possibilities, described in section 2.4, for deforming the cone on Y to a smooth G_2-manifold X. However, since the cone on $\mathbf{S}^1\times\mathbf{S}^1$ is singular, and we do not have any previous knowledge of the behavior of a D-brane with this type of singularity, in this example going from M-theory to Type IIA does not lead to any immediate understanding of the dynamics. It merely leads to a restatement of the questions. Everything we will say in section 4 could, indeed, be restated in terms of D-branes in $\mathbf{R}^6$ (with worldsheet disc instantons playing the role of membrane instantons). This example has also been examined in [38].

Our basic approach to establishing that $Y/U(1) = \mathbf{S}^5$ and $X/U(1) = \mathbf{R}^6$ will not use a cartesian coordinate description but what one might call "generalized polar coordinates" better adapted to rotational symmetry. For example, if we choose an orthogonal decomposition

$$\mathbf{R}^6 = \mathbf{R}^4 \oplus \mathbf{R}^2, \tag{3.7}$$

then using polar coordinates (r,θ) in $\mathbf{R}^2$ we can view $\mathbf{R}^6$ as being "swept out," under the $SO(2)$-action, by the one-parameter family of half five-spaces

$$\mathbf{R}^5_+(\theta) = (x,r), \qquad x\in\mathbf{R}^4, \qquad r\geq 0, \tag{3.8}$$

for fixed θ in $[0,2\pi]$. Alternatively, we can restrict θ to lie in $[0,\pi]$ and allow negative r, thus sweeping out $\mathbf{R}^6$ by copies of $\mathbf{R}^5$.

A very similar story applies if we choose a decomposition

$$\mathbf{R}^6 = \mathbf{R}^3 \oplus \mathbf{R}^3. \tag{3.9}$$

This decomposition can conveniently be viewed in complex notation as the decomposition of $\mathbf{C}^3$ into its real and imaginary parts: $z = x + iy$. Now, for $x \in \mathbf{R}^3$ with $|x| = 1$, we set

$$\mathbf{R}^4(x) = (tx, y) \text{ with } t \in \mathbf{R},\ y \in \mathbf{R}^3, \tag{3.10}$$

noting that $\pm x$ give the same four-space. As x varies, the $\mathbf{R}^4(x)$ sweep out $\mathbf{R}^6$.

The key observation about our seven-manifolds X is that they naturally contain families of six-manifolds or five-manifolds (depending on the case) so that the quotients by $U(1)$ can be naturally identified with the relevant polar description of $\mathbf{R}^6$. These manifolds (of dimension six and five) will all be real vector bundles over the two-sphere $\mathbf{S}^2$, with appropriate $U(1)$-action. As a preliminary we shall need to understand these actions and their associated quotients. The nontrivial statements about quotients that we need are all statements in four dimensions, so we begin there.

3.2. Some Four-Dimensional Quotients

We shall be interested in a variety of basic examples of four-manifolds with $U(1)$ action. The prototype is of course $\mathbf{R}^4 = \mathbf{C}^2$ with complex scalar action. The quotient is $\mathbf{R}^3$. If we identify $\mathbf{C}^2$ with the quaternions $\mathbf{H}$, as a flat hyperkähler manifold, then $U(1)$ preserves this structure and the associated hyperkähler moment map:

$$\vec{\mu} : \mathbf{R}^4/U(1) \cong \mathbf{R}^3 \tag{3.11}$$

identifies $\mathbf{R}^4/U(1)$ with $\mathbf{R}^3$. ($\vec{\mu}$ was defined in (3.3).) Note that $\mathbf{R}^4 = \mathbf{H}$ gets its hyperkähler structure from right quaternion multiplication, and the $U(1)$ action comes from an embedding $\mathbf{C} \to \mathbf{H}$ using left quaternion multiplication.

The map (3.11) gives a local model for $U(1)$ actions on more general four-manifolds M. If a fixed point P is isolated, then the linearized action at that point gives a decomposition of the tangent space

$$T_P = \mathbf{R}^2(a) \oplus \mathbf{R}^2(b). \tag{3.12}$$

where the factors denote invariant subspaces $\mathbf{R}^2 = \mathbf{C}$ on which $U(1)$ acts by $\lambda \to \lambda^a, \lambda \to \lambda^b$, with integers a, b. To specify the signs of a, b we have to pick orientations of the $\mathbf{R}^2$'s.

Then the basic result (3.11) implies that, near P, the quotient $M/U(1)$ will be a smooth three-manifold provided $|a| = |b|$. If $|a| = |b| = 1$ we are, with appropriate orientations, in the case considered in (3.11), while if $|a| = |b| = k$ then the $U(1)$ action factors through a cyclic group of order k and reduces back to (3.11) again.

If $b = 0$ and $|a| > 0$, then the local model is the action of $U(1)$ on $\mathbf{C}^2$ given by

$$\lambda\left(z_1, z_2\right) = \left(\lambda^a z_1, z_2\right). \tag{3.13}$$

The quotient by $U(1)$ is clearly $\mathbf{R}_+ \times \mathbf{R}^2 = \mathbf{R}^3_+$, a half-space with boundary $\mathbf{R}^2$. This applies locally to a four-manifold near a fixed surface with such weights (a, b).

With these two model examples, we now want to move on to consider the case when M is a complex line-bundle over $\mathbf{CP}^1 = \mathbf{S}^2$. Since

$$\mathbf{S}^2 = SU(2)/U(1), \tag{3.14}$$

a representation of $U(1)$ given by $\lambda \to \lambda^k$ defines a complex line-bundle H^k over $\mathbf{S}^2$, and moreover this has a natural action of $SU(2)$ on it. With the appropriate orientation of $\mathbf{S}^2$, the line-bundle H^k has first Chern class $c_1 = k$.

We shall also want to consider an additional $U(1)$-action on the total space of this line-bundle, given by multiplying by λ^n on each fibre. As a bundle with $SU(2) \times U(1)$ action, we shall denote this by $H^k(n)$.

Let F_1 be a $U(1)$ subgroup of $SU(2)$, F_2 the additional $U(1)$ that acts only on the fibres, and F the diagonal sum of F_1 and F_2. We want to look at the quotient of $H^k(n)$ by F. The fixed points of F are (for generic n and k) the points in the zero-section $\mathbf{S}^2$ that are fixed by F_1. In fact, F_1 has two fixed points, which we will call 0 and ∞. On the tangent planes to $\mathbf{S}^2$ at the two fixed points, F_1 acts with weights that are respectively 2 and -2. The 2 arises because $SU(2)$ is a double cover of $SO(3)$ (so a spinor on $\mathbf{S}^2$ transforms under F_1 with weight ± 1 and a tangent vector with weight ± 2). On the fibres of $H^k(n)$ over the two fixed points, F_1 acts with respective weights k and $-k$. [5] So its weights are $(2, k)$ at one fixed point and $(-2, -k)$ at the other. On the other hand, F_2 acts trivially on the tangent space to $\mathbf{S}^2$, and acts with weight n on the fibre. So it acts with weights $(0, n)$ at each fixed point.

[5] The wavefunction of an electron interacting with a magnetic monopole of charge k is a section of H^k. The minimum orbital angular momentum of such an electron is k, because a rotation acts on the fibre with weights k and $-k$.

Adding the weights of F_1 and those of F_2, we learn that the diagonal subgroup F acts at the tangent spaces to the fixed points with weights

$$(2, n+k), \qquad (-2, n-k). \tag{3.15}$$

The fixed points are isolated provided $|n| \neq |k|$. The quotient $H^k(n)/F$ will be a three-manifold provided

$$|n \pm k| = 2 \tag{3.16}$$

while if $|n| = |k| > 0$ the quotient will be a three-manifold with an $\mathbf{R}^2$-boundary.

We shall be interested, for application to our three seven-manifolds, in the quotients $H^k(n)/U(1)$ (with $U(1) = F$) in the following three special cases:

(I) In the first example, we will have $k = 2$, $n = 0$. Here, $H^2(0)$ is the tangent bundle over $\mathbf{S}^2$, with Chern class $c_1 = 2$ and with standard $U(1)$-action.

(II) In the second example, we will have $k = \pm n = 1$. $H^1(1)$ is the spin bundle ($c_1 = 1$), but with an additional $U(1)$ action by λ or λ^{-1} in the fibres.

(III) In the third example, we will have $k = 0$, $n = \pm 2$. $H^0(2)$ is the trivial bundle ($c_1 = 0$) with λ^2 or λ^{-2} action in the fibres.

From our general remarks earlier, we know that I and III lead to three-manifold quotients, whereas II leads to a three-manifold with boundary. We claim that these quotients are:

$$\begin{array}{ll} \mathrm{I} & H^2(0)/U(1) = \mathbf{R}^3 \\ \mathrm{II} & H^1(1)/U(1) = \mathbf{R}^3_+ \\ \mathrm{III} & H^0(2)/U(1) = \mathbf{R}^3. \end{array} \tag{3.17}$$

Note that changing the sign of n in case II or III simply amounts to switching the fixed points $0, \infty$ on the sphere and does not change the geometry. We shall therefore take $n > 0$.

Case I: $H^2(0)/U(1)$

$H^2(0)$ is the resolution of the A_1 singularity to a smooth hyperkähler manifold. The resolution of the A_1-singularity is the line bundle over $\mathbf{CP}^1$ with $c_1 = -2$ and (after changing the orientation to make $c_1 = +2$) the map from $H^2(0)/U(1)$ to $\mathbf{R}^3$ is given by the hyperkähler moment map μ_1 of the $U(1)$ action. This is a natural generalization of the prototype statement $\mathbf{R}^4/U(1) = \mathbf{R}^3$, which is really the $N = 0$ case of the A_N-singularity story.

The proof that $H^2(0)/U(1) = \mathbf{R}^3$ can be made completely explicit by writing the hyperkähler metric on $H^2(0)$ in the form [48]

$$ds^2 = U^{-1}(d\tau + \vec{\omega}\cdot d\vec{x})^2 + U d\vec{x}\cdot d\vec{x}, \quad \vec{x} \in \mathbf{R}^3, \quad U = \frac{1}{|\vec{x}-\vec{x}_0|} + \frac{1}{|\vec{x}-\vec{x}_1|}, \quad d\vec{\omega} = *dU. \quad (3.18)$$

$U(1)$ acts by translation of the angular coordinate τ, so dividing by $U(1)$ is accomplished by forgetting τ; the quotient is thus $\mathbf{R}^3$, parameterized by $\vec{x}$.

$H^2(0)$ has another $U(1)$ symmetry that commutes with the group $F = U(1)$ that we are dividing by, namely the group $F_2 = U(1)$ that acts only on the fibres of $H^2(0)$. F_2 corresponds to the rotation of $\mathbf{R}^3$ that leaves fixed the points $\vec{x}_0$ and $\vec{x}_1$.

Since the A_1 singularity is conical, its smooth resolution $H^2(0)$ is a particularly simple example of the process of deforming conical singularities that is the main theme of this paper, so it is worth examining the geometry here in more detail.

The map μ_1 collapses $\mathbf{S}^2$ (the zero-section) of $H^2(0)$ to an interval in $\mathbf{R}^3$ (the interval connecting $\vec{x}_0$ and $\vec{x}_1$), with $0, \infty$ becoming the end points. In fact the hyperkähler metric on $H^2(0) = T\mathbf{S}^2$ has a parameter r, essentially the radius of the sphere, and this becomes the length of the interval. Thus as $r \to 0$, $T\mathbf{S}^2$ degenerates to the cone on $\mathbf{S}^3/\mathbf{Z}_2 = \mathbf{RP}^3$ and the interval shrinks to a point. In the limit $r = 0$, we still find $\mathbf{R}^3$ but this time it is really got from dividing $\mathbf{C}^2/\mathbf{Z}_2$ by $U(1)$, or equivalently by dividing $\mathbf{C}^2$ by the $U(1)$ with weight $(2, 2)$. The origin in $\mathbf{R}^3$ really has multiplicity 2.

Thus, while the deformation of the cone on $\mathbf{RP}^3$ to a smooth four-manifold alters the topology, the corresponding deformation of the quotients by $U(1)$ does not. It simply involves expanding a point to an interval. Note however that the identification of $\mathbf{R}^3$ before and after the deformation involves a complicated stretching map in which spheres centred at the origin get stretched into ellipsoidal shapes surrounding the interval.

Similar remarks apply in fact to all the A_N-singularities.

Case II: $H^1(1)/U(1)$

We turn next to case II, the quotient $H^1(1)/U(1)$. The weights at 0 and ∞ are respectively $(2, 2)$ and $(-2, 0)$. Our aim is to construct a map

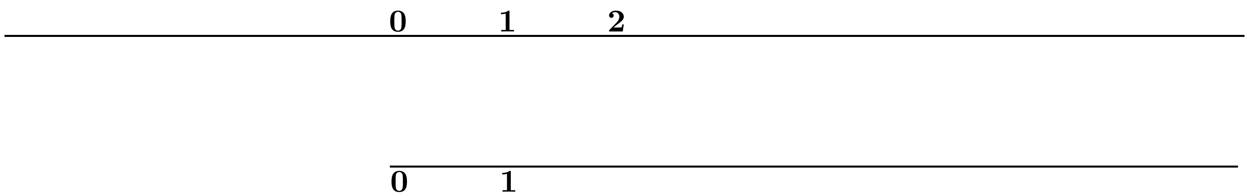
$$f : H^1(1) \longrightarrow \mathbf{R}^3_+ = (x, y, z) \in \mathbf{R}^3, \ z \geq 0 \quad (3.19)$$

which identifies the quotient by $U(1)$. The image of the zero-section will be contained in the half-line $x = y = 0$, and there will be rotational symmetry about this axis.

We shall briefly indicate two different proofs. The first is a "bare hands" identification with cutting and gluing, whereas the second is more elegant but more sophisticated.

The first argument uses, as models, two $U(1)$-actions on $\mathbf{R}^4$ whose quotients we have already discussed. For an action with weights $(-2, 0)$, the quotient is a half-space; this gives a local model near ∞. For an action with weights $(2, 2)$, the quotient is $\mathbf{R}^3$; this gives a local model near 0.

In the figure below, we sketch the ingredients in a direct analysis of the quotient $H^1(1)/U(1)$. The horizontal direction in the figure is the z direction; a transverse $\mathbf{R}^2$ is understood.



The first line is the image in $\mathbf{R}^3$ of three surfaces in $H^2(0)$. The interval $[0, 2]$ with 1 as midpoint represents the image of the 0-section of $H^2(0)$ with $\infty \to 0$ and $0 \to 2$; to the left and right of this interval are half-lines that are the images of the $\mathbf{R}^2$ fibres over $0, \infty$. Below this is a half-line that represents the quotient $\mathbf{R}^4/U(1)$ with weights $(-2, 0)$. In that half-line, the point 1 has no geometric significance, but is just chosen to match up with the top line.

We want the quotient $H^1(1)/U(1)$, rather than the quotient $H^2(0)/U(1)$ which is depicted in the top half of the figure. To convert $H^2(0)$ into $H^1(1)$, remove the half-line $z < 1$ in the top line and the half-line $z > 1$ in the bottom line, and glue the remaining parts at $z = 1$. In the four-manifolds we have, over this point, copies of $\mathbf{R}^2 \times U(1)$. We glue these together with a twist, using $U(1)$ to rotate $\mathbf{R}^2$.

It is not hard to see that we have, in this way, constructed $H^1(1)$. The fact that its quotient by $U(1)$ is a half-space now follows from our construction.

Our second proof uses again the fact that $H^2(0)/U(1) = \mathbf{R}^3$, but now we make the $H^2(0) = T\mathbf{S}^2$ depend on a parameter t, in such a way that in the limit $t \to 0$, the $\mathbf{S}^2$ splits up (by pinching at the equator) into two copies of $\mathbf{S}^2$. In this limit, $H^2(0)$ splits into two $H^1(1)$'s, and its quotient $\mathbf{R}^3$ splits into two copies of $\mathbf{R}^3_+$.

We start out with the projective plane $\mathbf{CP}^2$, with its standard line-bundle H having $c_1 = 1$, and we take $U(1)$ to act on the homogeneous coordinates of $\mathbf{CP}^2$ by

$$\lambda\,(z_1, z_2, z_3) = \left(z_1, \lambda^2 z_2, \lambda^{-2} z_3\right). \tag{3.20}$$

This has three isolated fixed points

$$A = (1,0,0), \quad B = (0,1,0), \quad C = (0,0,1). \tag{3.21}$$

$U(1)$ acts on the fibres of H over A, B, C with weights $(0, -2, 2)$. Restricting to the projective line AB (the copy of $\mathbf{CP}^1$ containing A and B), we find the line-bundle $H^1(1)$, and over AC we find $H^1(-1)$. These are the four-manifolds whose quotients by $U(1)$ should each be an $\mathbf{R}^3_+$.

Now we introduce the family of rational curves (i.e. two-spheres) with equation

$$z_2 z_3 = t z_1^2, \tag{3.22}$$

with t being our parameter. Note that each of these is invariant under our $U(1)$, and is in fact the closure of an orbit of the complexification $\mathbf{C}^*$. The bundle H over $\mathbf{CP}^2$ restricts on each of these rational curves to a copy of $H^2(0)$ (this follows from the weights at the fixed points B and C), so that by our previous analysis, quotienting out yields an $\mathbf{R}^3$ (depending on t). As $t \to 0$, these $\mathbf{R}^3$'s tend to the union of the quotients coming from the two branches of the degenerate curve $z_2 z_3 = 0$, i.e. the two lines AB, AC (note that the integers add up, so that $H^2(0)$ is the "sum" of $H^1(1)$ and $H^1(-1)$). By symmetry this must split the $\mathbf{R}^3$ into two half-spaces, as requested.

In fact, near A, the quotient of H (as a bundle over $\mathbf{CP}^2$) is a five-manifold and so there is little difficulty in checking the details of the limiting process.

Case III: $H^0(2)/U(1)$

We come finally to case III, the bundle $H^0(2)$, i.e. the product $\mathbf{S}^2 \times \mathbf{C}$ with $U(1)$ acting as usual on $\mathbf{S}^2$ (as the standard subgroup of $SU(2)$) and acting on $\mathbf{C}$ with weight 2. We shall show that $H^0(2)$ has the same quotient as $H^2(0)$ which, as we have seen, is

$\mathbf{R}^3$. To see this, we observe that both quotients can also be viewed as the quotient by the torus $U(1)^2$ acting on

$$\mathbf{S}^3 \times \mathbf{C} = (z_1, z_2, z_3) \ \text{with} \ |z_1|^2 + \left|z_2^2\right| = 1 \tag{3.23}$$

by

$$(\lambda, \mu)\,(z_1, z_2, z_3) = \left(\lambda z_1, \mu z_2, \lambda\mu^{-1} z_3\right). \tag{3.24}$$

Factoring out by the diagonal $U(1)$, $\lambda = \mu$, yields the product H^0, while factoring out by the anti-diagonal $\lambda = \mu^{-1}$ yields the non-trivial bundle H^2. The action of the remaining $U(1)$ identifies these two as $H^0(2)$ and $H^2(0)$. This completes the proof that

$$H^0(2)/U(1) = \mathbf{S}^3 \times \mathbf{C}/U(1)^2 = H^2(0)/U(1) = \mathbf{R}^3. \tag{3.25}$$

We could also have used a cutting and gluing argument as in case II.

Having dealt with these three cases of four-manifolds with $U(1)$ action, we are now ready to apply our results to the three cases of asymptotically conical G_2-manifolds with $U(1)$ action. All three are variations on the same theme, with minor differences. The four-manifold quotients I, II, III that we have just considered will arise in studying the similarly numbered G_2-manifolds.

3.3. Case I : $Y = \mathbf{CP}^3$

We begin with the case when $Y = \mathbf{CP}^3$ and X is the $\mathbf{R}^3$-bundle over $\mathbf{S}^4$ given by the anti-self-dual 2-forms. The action of $U(1)$ on $\mathbf{CP}^3$ is given in complex homogeneous coordinates by

$$\lambda\,(z_1, z_2, z_3, z_4) = \left(\lambda z_1, \lambda z_2, \lambda^{-1} z_3, \lambda^{-1} z_4\right). \tag{3.26}$$

It has two fixed $\mathbf{CP}^1$'s, which are of codimension 4. The action in the normal directions to the fixed point set has weights $(2, 2)$ or $(-2, -2)$. This ensures (as in section 3.1) that the quotient $\mathbf{CP}^3/U(1)$ is a (compact) five-manifold.

The fact that $\mathbf{CP}^3/U(1) = \mathbf{S}^5$ can be shown in a very elementary fashion. We can write a point in $\mathbf{S}^5$ in a unique fashion as a six-vector $(t\vec{x}, \sqrt{1-t^2}\vec{y})$ with $0 \le t \le 1$ and $\vec{x}, \vec{y} \in \mathbf{S}^2$. We normalize the z_i so $\sum_i |z_i|^2 = 1$, and rename them as (n_1, n_2, m_1, m_2). Then we simply define $t = \sqrt{|n_1|^2 + |n_2|^2}$, $\vec{x} = (n, \vec{\sigma} n)/(n, n)$, $\vec{y} = (m, \vec{\sigma} m)/(m, m)$, and this gives an isomorphism from $\mathbf{CP}^3/U(1)$ to $\mathbf{S}^5$.

If X is a cone on Y, its quotient by $U(1)$ is a cone on $\mathbf{S}^5$, or in other words $\mathbf{R}^6$.

We want to show that the smooth G_2-manifold X that is asymptotic to a cone on Y also obeys $X/U(1) = \mathbf{R}^6$. We recall that X is the bundle of anti-self-dual two-forms on $\mathbf{S}^4$. The $U(1)$ in (3.26) is a subgroup of the $Sp(2)$ symmetry group of $\mathbf{CP}^3$ (as described in section 2.2). Identifying $Sp(2) = Spin(5)$ as a symmetry group of $\mathbf{S}^4$, this particular $U(1)$ acts by rotation on two of the coordinates of $\mathbf{S}^4$. In other words, if we regard $\mathbf{S}^4$ as the unit sphere in $\mathbf{R}^5$, then there is a decomposition

$$\mathbf{R}^5 = \mathbf{R}^2 \oplus \mathbf{R}^3 \tag{3.27}$$

such that $U(1)$ is the rotation group of $\mathbf{R}^2$; we will use coordinates $\vec{x} = (x_1, x_2) \in \mathbf{R}^2$, $\vec{y} = (y_1, y_2, y_3) \in \mathbf{R}^3$. The fixed point set of $U(1)$ in $\mathbf{S}^4$ is a copy of $\mathbf{S}^2$ (the unit sphere in $\mathbf{R}^3$). Over one of the fixed points in $\mathbf{S}^4$, there is up to a real multiple just one $U(1)$-invariant anti-self-dual two-form (it looks like $\epsilon - *\epsilon$, where ϵ is the volume form of the fixed $\mathbf{S}^2$ and $*$ is the Hodge dual). A fixed point in X is made by choosing a fixed point in $\mathbf{S}^4$ together with an anti-self-dual two-form, so the fixed point set in X is $\mathbf{S}^2 \times \mathbf{R}$. Thus, for this example, we have justified the assertions in (3.5), (3.6).

The quotient $\mathbf{S}^4/U(1)$ is a closed three-disc D^3. Indeed, by a $U(1)$ rotation, we can map any point in $\mathbf{S}^4$ to $x_2 = 0$, $x_1 = \sqrt{1 - \vec{y}^2}$, so $\mathbf{S}^4/U(1)$ is parameterized by $\vec{y}$ with $|\vec{y}| \leq 1$. If we omit the fixed $\mathbf{S}^2$, then over the open three-disc $\mathring{D}^3$, the bundle of anti-self-dual two-forms is just the cotangent bundle of the disc. (In fact, if α is a one-form on the open disc, then on identifying the open disc with the set $x_1 > 0$, $x_2 = 0$, we map α to the anti-self-dual two-form $dx^1 \wedge \alpha - *(dx^1 \wedge \alpha)$.)

We think of $\mathring{D}^3$ as living in $\mathbf{R}^3$. Its tangent bundle is trivial and can naturally be embedded in

$$\mathbf{C}^3 = \mathbf{R}^3 + i\mathbf{R}^3 \tag{3.28}$$

with the disc embedded in $\mathbf{R}^3$, while $i\mathbf{R}^3$ represents the tangent directions. This is of course compatible with the natural action of $SO(3)$ on the disc (which in fact can be extended to $SO(3,1)$ using the conformal symmetries of $\mathbf{S}^4$).

At this stage one might think we are almost home, in that if $\mathring{X}$ is the part of X that lies over the open disc, we have identified

$$\mathring{X}/U(1) = T\mathring{D}^3 = \mathring{D}^3 \times \mathbf{R}^3, \tag{3.29}$$

and the right side is diffeomorphic to $\mathbf{R}^6$. However, this does not allow for the part of X that we have excised, namely the fibre of X over the fixed $\mathbf{S}^2$. The surprise is that, including this, we still get $\mathbf{R}^6$, with $\mathbf{R}^3 \times \mathring{D}^3$ as an open dense set.

To deal with this, we shall fix a point $u \in \mathbf{S}^2 \subset \mathbf{R}^3$, which we shall also identify with its image on the boundary of the closed disc D^3. Let I_u be the set of points in the closed disc of the form ru, $-1 \leq r \leq 1$, and let $\mathring{I}_u$ be the corresponding open interval with $|u| < 1$. Let X_u be the part of X that lies above I_u (so $\vec{y}$ is proportional to u), and let $\mathring{X}_u$ the open part of X_u (with $|\vec{y}| < 1$).

Restricting (3.29) to $\mathring{X}_u$, we get a "slice" of that fibration:

$$\mathring{X}_u/U(1) = \mathring{I}_u \times \mathbf{R}^3. \tag{3.30}$$

Our aim is to show that, on passing to the closure X_u of $\mathring{X}_u$, we get

$$X_u/U(1) = \mathbf{R}_u \oplus i\mathbf{R}^3, \tag{3.31}$$

where $\mathbf{R}_u$ is the line through u in $\mathbf{R}^3$. We will denote $\mathbf{R}_u \oplus i\mathbf{R}^3$ as $\mathbf{R}^4_u$. If (3.31) is true, then simply upon rotating u by $SO(3)$, the four-spaces $\mathbf{R}^4_u$ will sweep out $\mathbf{R}^6$ and establish the required identification

$$X/U(1) = \mathbf{R}^3 \oplus i\mathbf{R}^3 = \mathbf{C}^3. \tag{3.32}$$

We are thus reduced to establishing (3.30), and this is where our preliminary work on $U(1)$-quotients of four-manifolds will pay off. X_u is just the restriction of the $\mathbf{R}^3$ bundle X from $\mathbf{S}^4$ to the two sphere $\mathbf{S}^2_u$ which is cut out by the three space

$$\mathbf{R}^3_u = \mathbf{R}^2 \oplus \mathbf{R}_u \tag{3.33}$$

in the original decomposition (3.27) of $\mathbf{R}^5$. (In other words, $\mathbf{S}^2_u$ is characterized by $\vec{y}$ being a multiple of u.)

The bundle X_u over $\mathbf{S}^2_u$ is homogeneous, and we can identify it by looking at the representation of $U(1)$ at the point u. An anti-self-dual two-form at a point on $\mathbf{S}^2_u \subset \mathbf{S}^4$ that has an even number of indices tangent to $\mathbf{S}^2_u$ is invariant under rotations of $\mathbf{S}^2_u$ around that point, while those with one index tangent to $\mathbf{S}^2_u$ (and one normal index) transform as tangent vectors to $\mathbf{S}^2_u$, or equivalently as vectors in $\mathbf{R}^3$ that are perpendicular to u. So we get a decomposition

$$\Lambda^2_-(u) = \mathbf{R}u \oplus u^\perp. \tag{3.34}$$

This decomposition shows that, as a bundle over $\mathbf{S}^2_u$, the five-manifold X_u is

$$X_u = H^2(0)_u \times i\mathbf{R}u. \tag{3.35}$$

Hence, using the fact that $H^2(0)/U(1) = \mathbf{R}^3$, its quotient by $U(1)$ is

$$X_u/U(1) = \mathbf{R}^3_u \oplus i\mathbf{R}u, \tag{3.36}$$

and the factor $\mathbf{R}^3_u$ can naturally be identified as

$$\mathbf{R}^3_u = \mathbf{R}u \oplus iu^\perp \tag{3.37}$$

This implies (3.31), and hence (3.32).

The fixed point set $L = \mathbf{S}^2 \times \mathbf{R}$ in X gets mapped to the three-manifold in $\mathbf{R}^6$ given by the union of the lines $u + i\mathbf{R}u$, for $u \in \mathbf{R}^3, |u| = 1$. But as noted in [44], this is the natural embedding of the normal bundle to $\mathbf{S}^2$ in the tangent bundle

$$T\mathbf{R}^3 = \mathbf{R}^3 \oplus i\mathbf{R}^3 = \mathbf{C}^3, \tag{3.38}$$

and hence is Lagrangian. The radius of the $\mathbf{S}^2$ at the "center" of L is the modulus that is related to the radius of the $\mathbf{S}^4$ at the "center" of X.

3.4. Case II : $\mathrm{Y} = \mathrm{U}(3)/\mathrm{U}(1)^3$

We can now examine the second case, in which our six-manifold is $Y = SU(3)/U(1)^2 = U(3)/U(1)^3$, and, as in the previous case, X is an $\mathbf{R}^3$ bundle of anti-self-dual two-forms, this time over $\mathbf{CP}^2$. Y is just the unit sphere bundle in X, and the fibration $Y \to \mathbf{CP}^2$ is given in terms of groups by

$$U(3)/U(1)^3 \longrightarrow U(3)/U(1) \times U(2). \tag{3.39}$$

The $U(1)$-subgroup which we want to divide by to get to Type IIA is given by the left action of the scalars in $U(2)$, or in other words by $U(3)$ elements of the form

$$\begin{pmatrix} 1 & & \\ & \lambda & \\ & & \lambda \end{pmatrix}. \tag{3.40}$$

On $\mathbf{CP}^2$, this has an isolated fixed point $A = (1,0,0)$ and also a fixed $\mathbf{CP}^1$, which we will call B, on which the first coordinate vanishes. $U(1)$ acts trivially on the fibre X_A over A. At each point of B, the action of $U(1)$ decomposes the space of anti-self-dual two-forms into $\mathbf{R} \oplus \mathbf{R}^2$ with $\mathbf{R}$ fixed and $\mathbf{R}^2$ rotated. Thus our fixed three-manifold L in X this time consists of two components

$$L = \mathbf{R}^3 \cup \mathbf{S}^2 \times \mathbf{R}. \tag{3.41}$$

This cuts out on Y the fixed-point set

$$F = \mathbf{S}^2 \cup \mathbf{S}^2 \cup \mathbf{S}^2 \tag{3.42}$$

corresponding to the ends of L. Note that the Σ_3 symmetry of F, permuting the $\mathbf{S}^2$'s, is broken in L, where one of the $\mathbf{S}^2$'s has been preferred. This is in keeping with the discussion of the Σ_3 symmetry of Y in section 2.3.

We now proceed rather as in the previous example. For every $u \in B$, we denote as $\mathbf{S}^2_u$ the copy of $\mathbf{CP}^1$ that contains A and u. We view $\mathbf{CP}^2$ as built up from the two-parameter family of these $\mathbf{CP}^1$'s. Of course, the group $SO(3)$ (of rotations of B) acts on this family. Note that each $\mathbf{S}^2_u$ is acted on by the $U(1)$ in (3.40); its fixed points are the points A, u, corresponding to the points called $0, \infty$ in section 3.2.

Given this decomposition of $\mathbf{CP}^2$, we consider the corresponding decomposition of X into a family of five-manifolds X_u which lie above $\mathbf{S}^2_u$ in X. Again, under the action of $SO(3)$, they sweep out X with axis X_A, the fibre of X over A.

Each X_u is an $\mathbf{R}^3$ bundle over the two-sphere $\mathbf{S}^2_u$. The representation Λ^2_- of $SO(4)$ (consisting of anti-self-dual two-forms) when restricted to the subgroup $U(1) \subset U(2) \subset SO(4)$ that is given in $U(2)$ by

$$\lambda \longrightarrow \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \tag{3.43}$$

decomposes as

$$\mathbf{R}^3 = \mathbf{R} \oplus \mathbf{C} \tag{3.44}$$

where $U(1)$ acts with weight 1 on $\mathbf{C}$. It follows that X_u, as a bundle over $\mathbf{S}^2_u$, splits off a trivial factor $\mathbf{R}$ and leaves the standard complex line-bundle H over S^2. To emphasize the dependence on u, we shall write this decomposition as

$$X_u = H_u \times \mathbf{R}_u. \tag{3.45}$$

Note that the trivial factor $\mathbf{R}_u$ enables us to identify the relevant copies of $\mathbf{R} \subset \mathbf{R}^3$ over the two fixed points A and u. Over u, this $\mathbf{R}_u$ is the fibre of L over $u \in \mathbf{S}^2_u$, while over A it lies in the fixed $\mathbf{R}^3 = X_A$, which is independent of u. If we denote as $\mathbf{R}^3_B$ a copy of $\mathbf{R}^3$ which contains $B = \mathbf{S}^2$ as unit sphere, we can think of L as the normal bundle to $\mathbf{S}^2 \subset \mathbf{R}^3_B$. There is then a natural identification of $\mathbf{R}^3_B$ with $\mathbf{R}^3 = X_A$, which matches up the common factors $\mathbf{R}_u$, as u varies on B. In other words there is a natural identification

$$\mathbf{R}^3_B = X_A \tag{3.46}$$

compatible with the rotation action of $SU(2)$.

With this analysis of the geometry of X in terms of the family of X_u's, we now move on to consider the quotient by $U(1)$. First we note that the quotient $\mathbf{CP}^2/U(1)$ is the unit disc D in $\mathbf{R}^3_B$.[6] The center of D is the point A, and its boundary is $B = \mathbf{S}^2$. Each $\mathbf{S}^2_u$ projects to the corresponding radial interval, from the centre A to the boundary point u. To understand the quotient of $X_u = H_u \times \mathbf{R}_u$ by $U(1)$ we need to know the "twist" on H_u i.e., in the notation of section 3.2, which $H^k(n)$ we have. We already know that $k = 1$. We could determine n by carefully examining the group actions but, more simply, we can observe that, since the fixed point set over u in X_u is $\mathbf{R}_u$, and has codimension 4, the quotient $X_u/U(1)$ is a manifold around this point and this requires $n = \pm 1$. But at A we know that we get a manifold with boundary. This determines the sign of n, giving $n = -1$. Thus we can make (3.45) more precise by writing

$$X_u = H^1_u(-1) \times \mathbf{R}_u \tag{3.47}$$

We are therefore in case II of section 3.2, and the quotient

$$H^1_u(-1)/U(1) = \mathbf{R}^3_+(u) \tag{3.48}$$

is a half-space, depending on u. Its boundary lies in $\mathbf{R}^3 = X_A$ and is the orthogonal complement $\mathbf{R}^\perp_u$ of $\mathbf{R}_u$. Thus $X_u/U(1)$ is an $\mathbf{R}^4_+$, depending on u, which has the fixed $R^3 = X_A$ as boundary. If we write the right side of (3.48) as

$$\mathbf{R}^3_+(u) = \mathbf{R}^2(u) \oplus \mathbf{R}_+(u), \tag{3.49}$$

then the image of the zero-section of the bundle H_u is the unit interval in $\mathbf{R}_+(u)$. So we have to identify $\mathbf{R}_+(u)$ with the half line in $\mathbf{R}^3_B$ defined by $u \in \mathbf{S}^2$. The first factor $\mathbf{R}^2(u)$ in (3.49) is the boundary and we have already seen that this is $\mathbf{R}^\perp_u \subset X_A$.

To keep track of all this geometry we now introduce

$$\mathbf{R}^6 = \mathbf{C}^3 = \mathbf{R}^3 \oplus i\mathbf{R}^3 = \mathbf{R}^3_B \oplus X_A, \tag{3.50}$$

[6] This is proved by directly examining the action of $U(1)$ on the homogeneous coordinates of $\mathbf{CP}^2$. One describes $\mathbf{CP}^2$ with coordinates (z_1, z_2, z_3), normalized to $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. Dividing by $U(1)$ can be accomplished by restricting to $z_1 > 0$. So the quotient is parameterized by z_2, z_3 with $|z_2|^2 + |z_3|^2 \leq 1$.

using the identification (3.46). With this notation,

$$H^1(-1)_u/U(1) = \mathbf{R}u \oplus i(\mathbf{R}u)^{\perp} \tag{3.51}$$

and

$$X_u/U(1) = \mathbf{R}u \oplus i\mathbf{R}^3. \tag{3.52}$$

Rotating by $SU(2)$ acting diagonally on $\mathbf{R}^6 = \mathbf{C}^3$, (3.52) implies that

$$X/U(1) = \mathbf{C}^3, \tag{3.53}$$

as required. Moreover this is compatible with the $SO(3)$ symmetry.

The fixed point set L in X, consists as we have seen of two components, say L_0, L_1. Under the identification (3.53), we see that

$$\begin{aligned} L_0 &= i\mathbf{R}^3 \\ L_1 &= \text{union of all } u + i\mathbf{R}u, \text{ for } u \in \mathbf{S}^2 \subset \mathbf{R}^3. \end{aligned} \tag{3.54}$$

The component L_1 is the same one we met in case I and, as pointed out there, it is Lagrangian. The component L_0 is trivially Lagrangian.

3.5. Case III*:* $Y = \mathbf{S}^3 \times \mathbf{S}^3$

For the third case, we take

$$Y = SU(2)^3/SU(2) = \mathbf{S}^3 \times \mathbf{S}^3 \tag{3.55}$$

with $SU(2)^3$ and Σ_3 symmetry. We take our $U(1)$ subgroup of $SU(2)^3$ to be the diagonal subgroup (acting on the left). Identifying Y with (say) the product of the last two factors of $SU(2)$, this action becomes conjugation on each factor, with the fixed point set F being a torus

$$F = \mathbf{S}^1 \times \mathbf{S}^1. \tag{3.56}$$

Since the action in the normal direction is by complex scalars on $\mathbf{C}^2$, the quotient $Y/U(1)$ is again a (compact) five-manifold.

Our seven-manifold X is, as explained in section 2, an $\mathbf{R}^4$ bundle over $\mathbf{S}^3$ which is topologically a product. If we introduce the quaternions $\mathbf{H}$, with standard generators i, j, k and with $SU(2)$ being the unit quaternions then

$$\begin{aligned} Y =&(x, y) \text{ with } x, y \in H, \ |x| = |y| = 1 \\ X =&(x, y) \text{ with } x, y \in H, \ |x| = 1 \end{aligned} \tag{3.57}$$

and $U(1) \subset SU(2)$ is given by the embedding $\mathbf{C} \to \mathbf{H}$, acting on x and y by conjugation.

The action of $U(1)$ on X, Y is by conjugation and the fixed-points are then the pairs (x, y) with $x, y \in \mathbf{C}$. The fixed-point set L in X is just

$$L = \mathbf{S}^1 \times \mathbf{R}^2, \tag{3.58}$$

coming from points with $x, y \in \mathbf{C}$ and $|x| = 1$.

The quotient $\mathbf{S}^3/U(1)$ is a disc[7]

$$\mathbf{S}^3/U(1) = D^2 \subset \mathbf{C}, \tag{3.59}$$

which we can think of as the unit disc in the complex x-plane with the unit circle $\mathbf{S}^1$ coming from the fixed points. As observed in section 2, this is the main difference between cases I, II on one hand and case III on the other. Here we get the two-disc rather than the three-disc.

As before, we pick a point $u \in \mathbf{S}^1$, on the boundary of the disc, and consider the corresponding $U(1)$-invariant two-sphere $\mathbf{S}^2_u$ through u.[8] This projects onto the unit interval $[-u, u]$ in (3.59). Let X_u be restriction of X to S^2_u. Since X is a product bundle, this is just

$$X_u = \mathbf{S}^2_u \times \mathbf{R}^4 \tag{3.60}$$

and $U(1)$ acts on $\mathbf{R}^4$ by conjugation of quaternions, so that it decomposes into

$$\mathbf{R}^4 = \mathbf{C}(0) \oplus \mathbf{C}(2) \tag{3.61}$$

where the integer denotes the weight of the representation of $U(1)$. Thus, in the notation of section 3.2

$$X_u = H^0_u(2) \times \mathbf{R}^2. \tag{3.62}$$

Note, that unlike cases I, II, the trivial factor $\mathbf{R}^2$ here does not depend on u.

Now in section (3.4) we showed that

$$H^0_u(2)/U(1) = \mathbf{R}^3_u. \tag{3.63}$$

[7] Describing $\mathbf{S}^3$ by an equation $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$, $U(1)$ acts by rotations of the x_2-x_3 plane; the quotient by $U(1)$ can be taken by setting $x_3 = 0$ and $x_2 > 0$.

[8] In the notation of the last footnote, u is of the form $(x_0, x_1, 0, 0)$, and $\mathbf{S}^2_u$ consists of points whose first two coordinates are a multiple of u.

where here there is a dependence on u. From (3.62), it follows that

$$X_u/U(1) = \mathbf{R}^3_u \times \mathbf{R}^2 \tag{3.64}$$

The line in $\mathbf{R}^3_u$ which contains the image of the zero-section can naturally be identified with the line $\mathbf{R}u \subset \mathbf{C}$ in $D^2 = \mathbf{S}^3/U(1)$, and the orthogonal $\mathbf{R}^2$ can be identified with the $\mathbf{C}(2)$ factor in (3.61). Thus

$$X_u/U(1) = \mathbf{R}^4 \oplus \mathbf{R}u \subset \mathbf{R}^4 \oplus \mathbf{R}^2 = \mathbf{R}^6. \tag{3.65}$$

Finally, rotating u in the complex plane leads to the desired identification

$$X/U(1) = \mathbf{R}^6. \tag{3.66}$$

The fixed-point set in X becomes the subspace

$$L = \mathbf{C}(0) \times \mathbf{S}^1, \tag{3.67}$$

when $\mathbf{C}(0)$ is the first factor in (3.61).

So far we have ignored the extra symmetries of the situation, but in fact $U(1)^3$, modulo the diagonal, acts on $X/U(1)$ and hence by (3.66) on $\mathbf{R}^6$. From (3.61) and (3.65), we already have a decomposition of $\mathbf{R}^6$ into three copies of $\mathbf{R}^2$. For the right orientations, identifying each $\mathbf{R}^2$ with a copy of $\mathbf{C}$, a calculation shows that (3.66) is compatible with the $U(1)^3$ action provided $(\lambda_1, \lambda_2, \lambda_3) \in U(1)^3$ acts on the three factors by

$$\lambda_2\lambda_3^{-1},\ \lambda_3\lambda_2^{-1},\ \lambda_1\lambda_2^{-1}. \tag{3.68}$$

3.6. The Lebrun Manifolds

Cases I and II have much in common. In each case, the seven-manifold is the $\mathbf{R}^3$-bundle of anti-self-dual two-forms over a compact four-manifold M. Moreover, there is a $U(1)$-action for which the quotient is a three-disc, $M/U(1) = D^3$, with the boundary $\mathbf{S}^2$ of D^3 arising from a component of the fixed-point-set of the $U(1)$-action. The difference between cases I and II is just that this $\mathbf{S}^2$ is the whole fixed-point-set in case I, while in case II there is also an isolated fixed-point which gives a distinguished point interior to D^3. Finally the sphere bundle Y of X is a complex manifold, being the twistor space of a self-dual conformal structure on M.

There is actually a whole sequence of four-manifolds $M(n)$ which share all these properties, so that

$$
\begin{aligned}
&M(0) = \mathbf{S}^4 \quad \text{(case I)} \\
&M(1) = \mathbf{CP}^2 \quad \text{(case II)} \\
&M(n) = \mathbf{CP}^2 \# \mathbf{CP}^2 \# \ldots \# \mathbf{CP}^2 \quad (n \text{ times}),
\end{aligned}
\tag{3.69}
$$

where # denotes the operation of "connected sum". This means that we excise small balls and attach the remaining manifolds by small tubes, in the same way as (in dimension 2) a surface of genus g is a connected sum of tori.

The manifolds $M(n)$ were studied by Lebrun [49] and we shall refer to them as Lebrun manifolds. We want to explore the possibility of deriving from them M-theory duals to more general brane configurations in $\mathbf{R}^6$. We begin by reviewing their construction and properties.

In section 3.2 we recalled the Gibbons-Hawking ansatz for constructing the ALE manifolds of type A. We gave in eqn. (3.18) the formula for A_1. The general case is precisely similar, and it can be generalized further by introducing a "mass parameter," taking the harmonic function U on $\mathbf{R}^3$ to be

$$
U = c + \sum_{i=1}^{n} \frac{1}{|\vec{x} - \vec{x}_i|}, \tag{3.70}
$$

where $c \geq 0$. If $c = 0$, we get the ALE manifolds, while if $c \neq 0$, we get the ALF manifolds, their Taub-NUT counterparts, in which the four-manifold is locally asymptotic to the product $\mathbf{R}^3 \times \mathbf{S}^1$. The parameter c is inverse to the radius of the circle factor so that, as c tends to 0, the circle becomes a copy of $\mathbf{R}$.

All this can be generalized with $\mathbf{R}^3$ replaced by the hyperbolic three-space H^3, of constant curvature -1, to give a complete Riemannian four-manifold M_n. We simply replace the factor $1/|\vec{x} - \vec{x}_i|$ by the corresponding Green's function for the hyperbolic metric. For small distances, the metric approximates the flat metric, so the behaviour near the points x_i is the same as for $\mathbf{R}^3$. For $x \to \infty$, we get a metric locally asymptotic to $H^3 \times \mathbf{S}^1$. Topologically the four-manifolds M_n are the same as in the Euclidean case, i.e. they are the resolution of the A_{n-1} singularity.

If, in particular, we take $c = 1$, then M_0 is conformally flat and by adding a copy of a two-sphere to it at infinity, one makes $\mathbf{S}^4$.[9] More generally, for $c = 1$, the four-manifold M_n looks at infinity just like M_0 (as $U \to 1$ at infinity for all n), so it can be conformally compactified at infinity by adding a copy of $\mathbf{S}^2$, to give a compact manifold $M(n)$. Note that the special value $c = 1$ is linked to the fact that we took the curvature κ of H^3 to be -1: in general we would take $c^2\kappa = -1$.

The Lebrun manifolds $M(n)$ have, from their construction, the following properties:

(A) $M(n)$ is conformally self-dual.

(B) $M(n)$ has a $U(1)$-action (acting by translations of τ just as in eqn. (3.18)) with a fixed $\mathbf{S}^2$ and n isolated fixed points $x_1, \dots, x_n$, so that $M(n)/U(1) = D^3$.

In *(B)*, the fixed points x_i give n distinguished interior points of D^3.

We now want to describe a further property of the Lebrun manifolds $M(n)$. We pick a point, which will be denoted as ∞, in the fixed $\mathbf{S}^2$. We denote $M(n)$ with ∞ deleted as $M_\infty(n)$. For $n = 0$ we have the natural identifications, compatible with $U(1)$,

$$\begin{aligned} M_0 &= \mathbf{S}^4 - \mathbf{S}^2 = H^3 \times \mathbf{S}^1 = \mathbf{C} \times \mathbf{C}^* \\ M_\infty(0) &= \mathbf{S}^4 - \infty = \mathbf{C}^2 \end{aligned} \tag{3.71}$$

These involve the identification of H^3 with the upper-half 3-space $\mathbf{C} \times \mathbf{R}_+$ of pairs $(u, |v|)$ with u, v complex numbers and v non-zero.

Lebrun shows that $M_\infty(n)$ has the following further property

(C) There is a map $\pi : M_\infty(n) \to M_\infty(0) = \mathbf{C}^2$ compatible with the $U(1)$-action, which identifies $M_n(\infty)$ with the blow-up of $\mathbf{C}^2$ at the n points $\pi(x_i)$, which are of the form $(u_i, 0) \in \mathbf{C}^2$.

π takes the fixed surface in $M_\infty(n)$ into the line $v = 0$ in $\mathbf{C}^2$. The orientation of $M_\infty(n)$ that we will use is the opposite of the usual orientation of $\mathbf{C}^2$.

For $n = 1$, *(C)* gives the well-known fact that $\overline{\mathbf{CP}}^2$ (that is, $\mathbf{CP}^2$ with the opposite orientation) is the one-point compactification of the blow-up of a point in $\mathbf{C}^2$. The change of orientation reverses the sign of the self-intersection of the "exceptional line" (inverse

[9] To see that $\mathbf{S}^4$ minus a two-sphere is $H^3 \times \mathbf{S}^1$, note that the conformal group $SO(5,1)$ of $\mathbf{S}^4$ contains a subgroup $SO(3,1) \times SO(2)$ where $SO(2)$ acts by rotation of two of the coordinates; throwing away the fixed point set of $SO(2)$, which is a copy of $\mathbf{S}^2$, the rest is a homogeneous space of $SO(3,1) \times SO(2)$ which can be identified as $H^3 \times \mathbf{S}^1$. Of course $\mathbf{S}^4$ is conformally flat, so $H^3 \times \mathbf{S}^1$ is also.

image of the blown-up point), turning it from -1 into $+1$, and so agreeing with the self-intersection of a line in $\mathbf{CP}^2$. More generally, *(C)* implies the assertion made in (3.69) about the topology of $M(n)$.

It is perhaps worth pointing out that the metric on M_n, given by using the function U of eqn. (3.70), depends on the n points $x_1, \ldots, x_n$ of H^3. But the complex structure of M_n, as an open set of $M_\infty(n)$, given by *(C)*, depends only on the points $\pi(x_i)$. (Intrinsically, $\pi(x_i)$ is the other end of the infinite geodesic from ∞ to x_i in H^3.) Moreover, the complex structure of M_n varies with the choice of the point ∞ in the fixed $\mathbf{S}^2$. This is very similar to the story of the complex structures on the hyperkähler manifolds of type A_n. Although the metric on M_n is not hyperkähler, Lebrun shows that it is kähler with zero scalar curvature (this guarantees the conformal self-duality for the other orientation).

Having summarized the properties of the Lebrun manifolds $M(n)$, we are now ready to move on to our bundles $X(n)$ and $Y(n)$ over them. As before, $X(n)$ is the bundle of anti-self-dual two-forms and $Y(n)$ the associated sphere bundle. Note that the notion of anti-self-duality makes sense for a conformal structure, not just for a metric, and so it makes sense for the conformal compactification $M(n)$ of M_n (actually Lebrun does give an explicit metric in the conformal class). $Y(n)$ is the associated sphere bundle and, being the twistor space of $M(n)$, it has a natural complex structure (though we shall not use this fact).

The action of $U(1)$ on $M(n)$ then extends naturally to actions on $X(n)$ and $Y(n)$. The behaviour at the fixed points in $M(n)$ follows from that in the model example $n = 1$, and gives fixed manifolds $F(n)$ inside $Y(n)$ and $L(n)$ inside $X(n)$ as follows:

$$F(n) = \mathbf{S}^2 \cup \mathbf{S}^2 \cup \ldots \cup \mathbf{S}^2 \tag{3.72}$$

$$L(n) = \mathbf{R}^3 \cup \mathbf{R}^3 \cup \ldots \cup \mathbf{R}^3 \cup (\mathbf{S}^2 \times \mathbf{R}). \tag{3.73}$$

In (3.72), there are $n + 2$ copies of $\mathbf{S}^2$, while in (3.73) there are n copies of $\mathbf{R}^3$. Since the n copies of $\mathbf{R}^3$ are disjoint, the corresponding n copies of $\mathbf{S}^2$ in (3.72) are unlinked. However, the last 2 copies of $\mathbf{S}^2$ are linked, and each is linked to the first n copies. This all follows from the case of $n = 1$. This means that the singular seven-manifold of which $X(n)$ is a deformation is not the cone on $Y(n)$ for $n \geq 2$. Instead it has a singularity just like that for $n = 1$, with just one of the first n unlinked $\mathbf{S}^2$ being shrunk to a point, together with the last two. Thus the local story does not change when we increase n. If we take a maximally degenerated Lebrun manifold, with the $\vec{x}_i$ all equal, it appears that the n disjoint $\mathbf{R}^3$'s become coincident, so the branes will consist of three copies of $\mathbf{R}^3$ of multiplicity $(1,1,n)$, rather as in the situation we consider in section 3.7 below.

We now want to show that

$$\begin{aligned} Y(n)/U(1) &= \mathbf{S}^5 \\ X(n)/U(1) &= \mathbf{R}^6, \end{aligned} \tag{3.74}$$

extending the results for $n=0$ (Case I) and $n=1$ (Case II). Since for $n>1$ we no longer have $SO(3)$-symmetry, we cannot, as before, reduce the problem to one in four dimensions. Instead we shall use the connected sum decomposition (3.69), together with the special cases $n=0,1$ already established.

Before embarking on the details, we should explain our strategy. As an analogue, recall that the connected sum of two n-spheres is still an n-sphere. This can be seen most explicitly if we bisect along an equator and then remove the southern hemisphere of one and the northern hemisphere of the other. Gluing the remaining pieces we clearly get another n-sphere. There is a similar story if we replace the n-sphere by $\mathbf{R}^n$, bisecting it into two half-spaces. We shall show that the connected sum operation on the four-manifolds $M(n)$ leads essentially to this bisection picture on $\mathbf{R}^6$.

Since we want to keep track of the $U(1)$-action, we want the more precise description of (3.69) given by *(C)*. For our present purposes, since we are not interested in the complex structure, we can rephrase this as follows. Starting with $\mathbf{S}^4$ and its $U(1)$-action with $\mathbf{S}^2$ as fixed-point set, we choose n distinct points $x_1,\ldots,x_n$ on $\mathbf{S}^2$. Blowing up these we get $M(n)$. Topologically, the blowup means that we excise small $U(1)$-invariant balls V_i around each x_i, whose boundaries are three-spheres, and then replace them by a two-plane bundle W_i over $\mathbf{S}^2$ using the standard fibration $\mathbf{S}^3 \to \mathbf{S}^2$. Note that such a W is also the complement of a ball in $\mathbf{CP}^2$, so that the blow-up operation is indeed the same as forming the connected sum with $\mathbf{CP}^2$.

If we blow up just one point, this gives the manifold $M(1) = \mathbf{CP}^2$, so that this will provide the local model around each point. Each time we modify $\mathbf{S}^4$ near a point x we can describe the corresponding modification of $X(0)$. We remove $V \times \mathbf{R}^3$ (the part of $X(0)$ over V) and replace it by the $\mathbf{R}^3$ bundle over W, given by the local model.

Next we need to examine how these modifications behave for the quotients by $U(1)$. We have already shown that $X(0)/U(1) = \mathbf{R}^6$, and our explicit description in section 3.3 shows that the fibre $\mathbf{R}^3$ over x_i goes into a half-plane $H = \mathbf{R}^2_+$. The neighbourhood $V \times \mathbf{R}^3$ will therefore go into a neighbourhood U of H in $\mathbf{R}^6$. It is not hard to see that there is a diffeomorphism of the pair $(\mathbf{R}^6, U)$ into $(\mathbf{R}^6, \mathbf{R}^6_+)$, i.e we have "bisected" $\mathbf{R}^6$. The easiest way to verify this is to start with a bisection of $\mathbf{S}^4$, i.e. to choose the original neighbourhood V to be a hemisphere in $\mathbf{S}^4$. Then by symmetry $U = \mathbf{R}^6_+$ (see the last part of Case II in section 3.2). If we now take a gradient flow along the meridians shrinking towards the south pole x, then the hemispheres get shrunk to arbitrarily small size. This diffeomorphism of $\mathbf{S}^4$, which is compatible with the $U(1)$-action, induces the required diffeomorphism on $\mathbf{R}^6$. We shall refer to U as a standard neighbourhood of $\mathbf{R}^2_+$ in $\mathbf{R}^6$.

Consider now the case $n = 1$. As we saw in section 3.4, $X(1)/U(1) = \mathbf{R}^6$. Moreover the explicit nature of this identification again shows that the $\mathbf{R}^3$-fibre over a point x of the fixed $\mathbf{S}^2$ in $X(1)$ is a half-plane H. The local model near here is provided by the case $n = 0$ and so a neighbourhood $V \times \mathbf{R}^3$ in $X(1)$ will go into a standard neighbourhood U of H. But we have already seen that such a U gives a bisection of $\mathbf{R}^6$. It follows that the $\mathbf{R}^3$-bundle over the complement, i.e. over W in $X(1) = \mathbf{CP}^2$, goes into the other half of this bisection. This means that the decomposition of $\mathbf{CP}^2$ into two unequal parts, one being a ball and the other being W, induces the trivial bisection of the quotient $\mathbf{R}^6$.

This shows, inductively, that the process of "adding" $\mathbf{CP}^2$'s (by connected sum operations), induces on the quotients $M(n)/U(1)$, just the trivial operation on $\mathbf{R}^6$ of bisection and then gluing two halves together as outlined before. This establishes that these quotients are always $\mathbf{R}^6$, as claimed.

The corresponding statement for $Y(n)$ follows at the same time.

Our analysis of the topology of $X(n)$ and its quotients by $U(1)$ does show that these manifolds generalize Cases I and II of the previous sections (for $n = 0, 1$). It naturally raises the question as to whether, for all n, $X(n)$ has a complete metric with G_2 holonomy, with the corresponding fixed-point-set $L(n)$, given by (3.73), being a special Lagrangian in $\mathbf{R}^6$. It is in fact easy to find such a special Lagrangian: we just take the case for $n = 1$ given by Joyce and add $n - 1$ parallel copies of the $\mathbf{R}^3$ component.

All of the explicit constructions of explicit G_2 holonomy metrics have used the presence of large symmetry groups, and this will not work for $n > 1$. What is needed is some kind of gluing technique, perhaps on the lines of [50].

3.7. Another Generalization

Finally, we want to point out some simple generalizations of our discussion that may be of physical interest.

Let X be any of the seven-manifolds with $U(1)$ action such that $X/U(1) = \mathbf{R}^6$. Let $\mathbf{Z}_n$ be the subgroup of $U(1)$ consisting of the points of order dividing n, and let $X_n = X/\mathbf{Z}_n$. Obviously, $U(1)$ still acts on X_n, and $X_n/U(1) = X/U(1) = \mathbf{R}^6$.

X_n is an orbifold rather than a manifold. We want to focus on the case that the fixed point set of $\mathbf{Z}_n$ is the same as the fixed point set of $U(1)$. This will always be so for generic n. In our examples, it is true for all $n \geq 2$. This being so, X_n is an orbifold with a locus of A_{n-1} singularities that is precisely the fixed point set of the $U(1)$ action on X_n.

The fixed points in the $U(1)$ action on X_n are the same as in the action on X. So the reduction to a Type IIA model via the quotient $X_n/U(1)$ leads to branes that occupy the same set in $\mathbf{R}^6$ that we get from $X/U(1)$. The difference is that since the fixed points in X_n are A_{n-1} singularities, the branes in the X_n model have multiplicity n.

All this holds whether X is a smooth manifold of G_2 holonomy or is conical. In drawing conclusions, it is helpful to start with the conical case:

(I) If X is the cone on $\mathbf{CP}^3$, the brane configuration that is a Type IIA dual of X_n consists of two copies of $\mathbf{R}^3$ meeting at the origin, each with multiplicity n. The associated low energy theory is a $U(n) \times U(n)$ theory with chiral multiplets transforming as $(\mathbf{n}, \overline{\mathbf{n}})$. Deforming X to a smooth manifold of G_2 holonomy and thereby deforming $\mathbf{R}^3 \cup \mathbf{R}^3$ to $\mathbf{R} \times \mathbf{S}^2$, the $U(n) \times U(n)$ is broken to a diagonal $U(n)$.

(II) The case that X is a cone on $SU(3)/U(1)^2$ is similar. The Type IIA dual of X_n is a brane configuration consisting of three copies of $\mathbf{R}^3$ meeting at a point, all with multiplicity n. The low energy theory has $U(n)^3$ symmetry and chiral multiplets transforming as $(\mathbf{n}, \overline{\mathbf{n}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{n}, \overline{\mathbf{n}}) \oplus (\overline{\mathbf{n}}, \mathbf{1}, \mathbf{n})$. Calling the three fields Φ_1, Φ_2, and Φ_3, there is a superpotential $\mathrm{Tr}\,\Phi_1\Phi_2\Phi_3$, just as in the $n = 1$ case, and there are various possibilities of symmetry breaking.

(III) For X a cone on $\mathbf{S}^3 \times \mathbf{S}^3$, the dual of X_n is a brane of multiplicity n that is a cone on $\mathbf{S}^1 \times \mathbf{S}^1$. We can draw no immediate conclusions, as we have no knowledge of the dynamical behavior of such a brane.

Notice in cases (I) and (II) that if a singularity like that of X_n would appear in a manifold of G_2 holonomy, we would get a gauge theory with chiral fermions. This might be of physical interest. For example, in case (II) with $n = 3$, the $U(3)^3$ gauge group with the indicated representation for the chiral superfields is very closely related to the standard model of particle physics with one generation of quarks and leptons.

One might wonder if examples I and II can be further generalized. For example, could we extend case I so that the branes will be a pair of $\mathbf{R}^3$'s of respective multiplicities (m, n) for arbitrary positive integers m, n? Candidate manifolds can be suggested as follows, though we do not know if they admit metrics of G_2 holonomy.

To build $\mathbf{CP}^3$, we start with $\mathbf{S}^7$, parameterized by four complex variables $z_1, \ldots, z_4$ with $\sum_{i=1}^4 |z_i|^2 = 1$. Then we divide by a $U(1)$ group that acts by

$$z_i \to e^{i\theta} z_i, \; i = 1, \ldots, 4. \tag{3.75}$$

The quotient is $Y = \mathbf{CP}^3$. In section 3.3, we divided by a second $U(1)$ which acts by

$$(z_1, z_2, z_3, z_4) \to (e^{i\theta} z_1, e^{i\theta} z_2, z_3, z_4). \tag{3.76}$$

The quotient, as we have seen, is $\mathbf{S}^5$. Now we will use an argument that we have already used in section 3.2 in case III. If the plan is to divide $\mathbf{S}^7$ by $U(1) \times U(1)$ to get $\mathbf{S}^5$, we can divide first by an arbitrary $U(1)$ subgroup of $U(1) \times U(1)$, acting say by

$$(z_1, z_2, z_3, z_4) \to (e^{in\theta} z_1, e^{in\theta} z_2, e^{im\theta} z_3, e^{im\theta} z_4). \tag{3.77}$$

The quotient is a weighted projective space, $Y(n, m) = \mathbf{WCP}^3_{n,n,m,m}$. Then we divide by the "second" $U(1)$, and we will be left with $Y(n, m)/U(1) = \mathbf{S}^5$, since we have just looked at the quotient $Y/U(1) = \mathbf{S}^7/U(1) \times U(1)$ in a different way.

By the same argument, if $X(n, m)$ is a cone on $Y(n, m)$, then $X(n, m)/U(1) = X/U(1) = \mathbf{R}^6$. What branes appear in the Type IIA model derived in this way from $X(n, m)$? We may as well assume that n and m are relatively prime (as a common factor can be removed by rescaling θ in (3.77)). The fixed points of the "second" $U(1)$ are the points with $z_1 = z_2 = 0$ or $z_3 = z_4 = 0$. These consist of two copies of $\mathbf{S}^2$, so on passing to the cone the branes fill two copies of $\mathbf{R}^3 \subset \mathbf{R}^6$. The two components of the fixed point set are A_{m-1} and A_{n-1} singularities, respectively, so the branes have multiplicities m and n.

More generally, suppose we want multiplicities $(m, n) = r(a, b)$ where a and b are relatively prime. We do this by combining the two constructions explained above. We start with $X(a, b)$, which gives a model with multiplicities (a, b), and then we divide by the $\mathbf{Z}_r$ subgroup of the "second" $U(1)$. The last step multiplies the multiplicities by r, so the quotient $X_r(a, b) = X(a, b)/\mathbf{Z}_r$ leads to a model in which the brane multiplicities are (m, n).

An analogous construction can be carried out in case II.

4. Quantum Parameter Space Of Cone On $\mathbf{S}^3 \times \mathbf{S}^3$

4.1. Nature Of The Problem

In this section, we return to the problem of M-theory on a G_2 manifold that is asymptotic to a cone on $Y = \mathbf{S}^3 \times \mathbf{S}^3$. We have seen in section 2.5 that there are three G_2 manifolds X_i all asymptotic to the same cone. It has been proposed [8] that there is a smooth curve $\mathcal{N}$ of theories that interpolates between different classical limits corresponding to the X_i. We will offer further support for this, but first let us explain why it might appear problematical.

First we recall how we showed, in section 2.3, in a superficially similar problem, involving a cone on $SU(3)/U(1)^2$, that there were three distinct branches $\mathcal{M}_i$ of the quantum moduli space corresponding to three classical spacetimes. We showed that the quantum problem had a symmetry group $K = U(1) \times U(1)$, determined by the symmetries of the C-field at infinity, and that on the three different spacetimes, there are three different unbroken $U(1)$ subgroups. The classification of $U(1)$ subgroups of $U(1) \times U(1)$ is discrete, and an observer at infinity can determine which branch the vacuum is in by determining which $U(1)$ is unbroken.

In the case of a cone on $\mathbf{S}^3 \times \mathbf{S}^3$, there are no global symmetries associated with the C-field, but we can try to make a somewhat similar (though ultimately fallacious) argument using the periods of the C-field. An observer at infinity measures a flat C-field, as otherwise the energy would be infinite. A flat C-field at infinity takes values in $E = H^3(Y;U(1)) = U(1) \times U(1)$. But not all possibilities are realized. For unbroken supersymmetry, the C-field on X_i must be flat at the classical level, so it must take values in $H^3(X_i;U(1))$. There is a natural map from $H^3(X_i;U(1))$ to $H^3(Y;U(1))$ which amounts to restricting to Y a flat C-field on X_i. However, this map is not an isomorphism; not all flat C-fields on Y extend over X_i as flat C-fields. In fact, $H^3(X_i;U(1)) = U(1)$ is mapped to a rank one subgroup E_i of $H^3(Y;U(1))$.

For different i, the E_i are different. In fact, they are permuted by triality, just like the generators D_i of $H_3(Y;\mathbf{Z})$ that were found in section 2.5. (In fact, the D_i are Poincaré dual to the E_i.)

Thus, an observer at infinity can measure the C-field as an element of $H^3(Y;U(1))$ and – classically – will find it to belong to one of the three distinguished $U(1)$ subgroups E_i. By finding which subgroup the C-field at infinity belongs to, the classical observer can thereby generically determine which spacetime X_1, X_2, or X_3 is present in the interior. If this procedure is valid quantum mechanically, then the moduli space of theories has distinct branches $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$. Of course, even classically there is an exceptional possibility that the C-field belongs to more than one of the $U(1)$'s (in which case it must vanish); in this case, the measurement of the C-field does not determine the manifold in the interior. This might seem to be a hint that the $\mathcal{N}_i$ intersect at some exceptional points, rather as we found in the superficially similar model of section 2.3.

In this discussion, we have put the emphasis on measurements at infinity, where semiclassical concepts apply, since there is no useful way to describe measurements in the interior, where the quantum gravity effects may be big.

Let us contrast this with a more familiar situation in Type II superstring theory: the "flop" between the two small resolutions Z_1 and Z_2 of the conifold singularity of a Calabi-Yau threefold in Type II superstring theory. In this case, the Neveu-Schwarz two-form field B plays the role analogous to C in M-theory. The B-field periods takes values in $H^2(Z_1;U(1))$ or $H^2(Z_2;U(1))$. Both of these groups are isomorphic to $U(1)$, and in this case the two $U(1)$'s are canonically the same. That is because both Z_1 and Z_2 are asymptotic to a cone on a five-manifold B that is topologically $\mathbf{S}^2 \times \mathbf{S}^3$. The second Betti number of B is one, equal to the second Betti number of the Z_i, and $H^2(B;U(1)) = U(1)$. The restriction maps from $H^2(Z_i;U(1))$ to $H^2(B;U(1))$ are isomorphisms, so in fact the three groups $H^2(B;U(1))$, $H^2(Z_1;U(1))$, and $H^2(Z_2;U(1))$ are all naturally isomorphic. So by a measurement of the B-field period at infinity, one cannot distinguish the manifolds Z_1 and Z_2.

Since supersymmetry relates the Kahler moduli to the B-field periods, this leads to the fact that the Kahler moduli of Z_1 and Z_2 fit together in a natural way at the classical level. (Moreover, all this carries over to other small resolutions of Calabi-Yau threefolds, even when the second Betti number is greater than one.) In fact, each Z_i contains an exceptional curve that is a two-sphere $\mathbf{S}^2_i$; one can naturally think of $\mathbf{S}^2_2$ as $\mathbf{S}^2_1$ continued to negative area. (The last assertion is related to the signs in the isomorphisms mentioned in the last paragraph.) By contrast, in M-theory on a manifold of G_2 holonomy, the metric moduli are related by supersymmetry to the C-field periods. The fact that the classical C-field periods on the X_i take values in different groups E_i also means that there is no way, classically, to match up the metric moduli of the three manifolds X_i.

We can be more explicit about this. The metric modulus of X_i is the volume V_i of the three-sphere $Q_i \cong \mathbf{S}^3_i$ at the "center" of X_i. Each V_i, classically, takes values in the set $[0,\infty\}$ and so runs over a ray, or half-line. In a copy of $\mathbf{R}^2$ that contains the lattice Λ, these rays (being permuted by triality) are at $2\pi/3$ angles to one another. They do not join smoothly.

How can we hope nevertheless to find a single smooth curve $\mathcal{N}$ that interpolates between the X_i? We must find a quantum correction to the claim that the C-field period measured at infinity on X_i takes values in the subgroup E_i of $H^3(Y;U(1))$. It must be that the C-field period takes values that are very close to E_i if the volume V_i is large, but not close when V_i becomes small. Then, one might continuously interpolate from X_i to X_j, with the period taking values in E_i in one limit, and in E_j in the other.

Let us see a little more concretely what is involved in getting such a quantum correction. When $Y = \mathbf{S}^3 \times \mathbf{S}^3$ is realized as the boundary of X_i, the three-sphere D_i defined in section 2.5 is "filled in" – it lies at infinity in the $\mathbf{R}^4$ factor of $X_i = \mathbf{R}^4 \times \mathbf{S}^3$. So, for a flat C-field, with $G = dC$ vanishing, we have

$$\int_{D_i} C = \int_{\mathbf{R}^4} G = 0. \tag{4.1}$$

Classically, we impose $G = 0$ to achieve supersymmetry. Quantum mechanically, we consider fluctuations around the classical supersymmetric state. If quantum corrections modify the statement $\int_{D_i} C = 0$, this would correct the statement that the C-field periods lie in E_i, and perhaps enable a smooth interpolation between the different classical manifolds X_i. As we will show in section 4.4, perturbative quantum corrections do not modify the statement that $\int_{D_i} C = 0$, but membrane instanton corrections do modify this statement.

In section 4.2, we will interpret the C-field periods as the arguments of holomorphic functions on $\mathcal{N}$. By exploiting the existence of those functions, we will, in section 4.3, making a reasonable assumption about a sense in which the X_i are the only classical limits of this problem, argue that $\mathcal{N}$ *must* have just one branch connnecting the X_i and give a precise description of $\mathcal{N}$. In section 4.4, we analyze the membrane instanton effects and show that they give the requisite corrections to $\mathcal{N}$. Details of the solution found in section 4.3 will be compared in section 5 to topological subtleties concerning the C-field.

4.2. Holomorphic Observables

The C-field periods on Y must be related by supersymmetry to some other observables that can be measured by an observer at infinity. Supersymmetry relates fluctuations in the C-field to fluctuations in the metric, so these other observables must involve the metric.

We are here considering C-fields that are flat near infinity, so that if D_i is any of the cycles at infinity discussed at the end of section 2.5, the periods

$$\int_{D_i} C \tag{4.2}$$

are independent of the radial coordinate r. Since the radius of D_i is proportional to r, this means that the components of C are of order $1/r^3$. So supersymmetry relates the C-field to a metric perturbation that is of relative order $1/r^3$, compared to the conical metric.

Moreover, a flat C-field preserves supersymmetry, so a perturbation of C that preserves the flatness is related to a perturbation of the metric that preserves the condition for G_2 holonomy. To find the metric perturbations that have this property, let us examine more closely the behavior of the metric (2.23) near $r = \infty$. We introduce a new radial coordinate y such that $dy^2 = dr^2/(1-(r_0/r)^3)$. To the accuracy that we will need, it suffices to take

$$y = r - \frac{r_0^3}{4r^2} + O(1/r^5). \tag{4.3}$$

The metric is then

$$ds^2 = dy^2 + \frac{y^2}{36}\Big(da^2 + db^2 + dc^2 - \frac{r_0^3}{2y^3}\left(f_1\, da^2 + f_2\, db^2 + f_3\, dc^2\right) + O(r_0^6/y^6)\Big) \tag{4.4}$$

with $(f_1, f_2, f_3) = (1, -2, 1)$. If we set $r_0 = 0$, we get the conical metric with the full Σ_3 symmetry. This is valid near $y = \infty$ even if $r_0 \neq 0$. Expanding in powers of r_0/y, the first correction to the conical metric is proportional to $(r_0/y)^3$ and is given explicitly in (4.4).

Obviously, if we make a cyclic permutation of a, b, c, this will cyclically permute the f_i. So a metric of the form in (4.4) with (f_1, f_2, f_3) equal to $(1, 1, -2)$ or $(-2, 1, 1)$ also has G_2 holonomy, to the given order in r_0/y. Moreover, since the term of lowest order in r_0/y obeys a *linear equation*, namely the linearization of the Einstein equation around the cone metric (nonlinearities determine the terms of higher order in r_0/y), we can take linear combinations of these solutions if we are only interested in the part of the metric of order $(r_0/y)^3$. Thus, the metric has G_2 holonomy to this order in r_0/y if (f_1, f_2, f_3) are taken to be any linear combination of $(1, -2, 1)$ and its cyclic permutations. Hence, G_2 holonomy is respected to this order if the coefficients f_i obey the one relation

$$f_1 + f_2 + f_3 = 0. \tag{4.5}$$

A flat C-field at infinity has periods

$$\alpha_i = \int_{D_i} C \mod 2\pi. \tag{4.6}$$

Actually, there is a subtlety in the definition of the α_i, because of a global anomaly in the membrane effective action; we postpone a discussion of this to section 5. Since $D_1 + D_2 + D_3 = 0$ in homology, as we explained in section 2.5, one would guess that $\alpha_1 + \alpha_2 + \alpha_3 = 0 \bmod 2\pi$, but for reasons we explain in section 5, the correct relation is

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi \bmod 2\pi. \tag{4.7}$$

Our discussion in section 4.1 started with the fact that an observer at infinity can measure the α_i. Now we can extend this to a supersymmetric set of measurements: the observer at infinity can also measure the f_i.

A classical physicist at infinity would expect the f_i to be a positive multiple of $(1, 1, -2)$ or a cyclic permutation thereof, and would expect one of the α_i to vanish. The reason for this is that classically, while one can obey the Einstein equations near infinity with any set of f_i that sum to zero, to obey the nonlinear Einstein equations in the interior and get a smooth manifold X_i of G_2 holonomy, the f_i must be of the form found in (4.3), or a cyclic permutation thereof. (See [40] for an analysis of the equations.) Likewise, on any X_i, the corresponding cycle D_i is contractible, and so α_i must vanish.

Let us consider the action of the permutation group Σ_3. The f_i and α_i are cyclically permuted under a cyclic permutation of a, b, c. Under the flip $(a, b, c) \to (c^{-1}, b^{-1}, a^{-1})$, we have $(f_1, f_2, f_3) \to (f_3, f_2, f_1)$, but $(\alpha_1, \alpha_2, \alpha_3) \to (-\alpha_3, -\alpha_2, -\alpha_1)$. The reason for the sign is that the flip reverses the orientation of Y, and this gives an extra minus sign in the transformation of C. So the holomorphic combinations of the f_i and α_j must be, for some constant k (we indicate later how k could be computed), $kf_1 + i(\alpha_2 - \alpha_3)$, $kf_2 + i(\alpha_3 - \alpha_1)$, and $kf_3 + i(\alpha_1 - \alpha_2)$. These combinations are mapped to themselves by Σ_3. Since the α_i are only defined modulo 2π, it is more convenient to work with combinations such as

$$y_i = \exp(kf_i + i(\alpha_{i+1} - \alpha_{i-1})). \tag{4.8}$$

The y_i, however, do not quite generate the ring of holomorphic observables. We can do better to define

$$\eta_i = \exp((2k/3)f_{i-1} + (k/3)f_i + i\alpha_i). \tag{4.9}$$

(So $\eta_i = (y_{i-1}^2 y_i)^{1/3}$.) Any holomorphic function of the f's and α's that is invariant under 2π shifts of the α's can be expressed in terms of the η's. The η_i are not independent; they obey

$$\eta_1\eta_2\eta_3 = -1. \tag{4.10}$$

Each η_i can be neither 0 nor ∞ without some of the f_i diverging to $\pm\infty$. So at finite points in the moduli space, the η_i take values in $\mathbf{C}^*$ (the complex plane with the origin omitted), and because of the constraint (4.10), the η_i taken together define a point in $W = \mathbf{C}^* \times \mathbf{C}^*$.

Let us verify that in the classical approximation, $\mathcal{N}$ is a holomorphic curve in W. On the branch of $\mathcal{N}$ corresponding to the manifold X_2, the f_i are in the ratio $(1,-2,1)$. Moreover, on this manifold, $\alpha_2 = 0$. Altogether, $\eta_2 = 1$, and hence, given (4.10), $\eta_1\eta_3 = -1$. These conditions define a holomorphic curve in W, as expected. More generally, on the branch corresponding to the classical manifold X_i one has, in the classical approximation,

$$\eta_i = 1, \quad \eta_{i-1}\eta_{i+1} = -1. \tag{4.11}$$

In the classical description, $\mathcal{N}$ consists of those three distinct branches.

In what limit is classical geometry valid? To see the manifold X_i semiclassically, its length scale r_0 must be large. In the limit $r_0 \to \infty$, we have $f_{i\pm 1} \to +\infty$ and $f_i \sim -2f_{i\pm 1}$, so

$$\eta_{i-1} \to \infty, \quad \eta_{i+1} \to 0. \tag{4.12}$$

We will argue later that, in fact, η_{i-1} has a simple pole and η_{i+1} a simple zero as $r_0 \to \infty$.

Let us record how the symmetries of the problem act on the η's. A cyclic permutation in Σ_3 permutes the η's in the obvious way, while the flip $(a,b,c) \to (c^{-1},b^{-1},a^{-1})$ acts by

$$(\eta_1,\eta_2,\eta_3) \to (\eta_3^{-1},\eta_2^{-1},\eta_1^{-1}). \tag{4.13}$$

There is one more symmetry to consider; as explained at the beginning of section 2.1, a reflection in the first factor of the spacetime $\mathbf{R}^4 \times X$ exchanges chiral and antichiral fields; it reverses the sign of the C-field while leaving fixed the metric parameters f_i, so it acts antiholomorphically, by

$$\eta_i \to \overline{\eta}_i. \tag{4.14}$$

The f_i have an intuitive meaning as "volume defects." Let us recall that in section 2.5, we defined three-cycles $D_i \cong \mathbf{S}^3$ in Y. D_1 was defined by the conditions

$$a = 1 = bc. \tag{4.15}$$

The others are obtained by cyclic permutation. We can embed D_1 in any of the X_i by setting, in addition, the radial coordinate y to an arbitrary constant. If we do so, D_1 has a y-dependent volume that behaves for $y \to \infty$ as

$$\frac{2\pi^2 y^3}{27} + \frac{\pi^2 r_0^3 f_1}{36} + O(r_0^6/y^3). \tag{4.16}$$

Thus, subtracting the divergent multiple of y^3, there is a finite volume defect $\pi^2 r_0^3 f_1/36$ at infinity. Likewise, all the D_i have volume defects $\pi^2 r_0^3 f_i/36$. The fact that for $r_0 \to \infty$, up to a cyclic permutation, the f_i are a positive multiple of $(1, 1, -2)$ means that the volume defects are also a positive multiple of $(1, 1, -2)$, and in particular precisely one of them is negative. This fact has an intuitive meaning. In the "interior" of X_i, precisely one of the D's, namely D_i, is "filled in" and has its volume go to zero. This is the one whose volume defect at infinity is negative. There is no smooth manifold of G_2 holonomy in which the volume defects are a *negative* multiple of $(1, 1, -2)$, roughly since there is no way to make a smooth manifold by filling in two of the D's.

4.3. Quantum Curve

To understand supersymmetric dynamics via holomorphy, one must understand the singularities. In the present case, a singularity arises when some of the f_i diverge to $\pm\infty$, and hence some η_i have zeroes or poles. In our definition of $\mathcal{N}$, we will include points where there are such zeroes or poles, so our $\mathcal{N}$ is really a compactification of the moduli space of coupling parameters.

For f_i to diverge to $\pm\infty$ means that the volume defects are diverging. It is reasonable to expect that such behavior can be understood classically. In the present problem, we will assume that the only way to get a zero or pole of the η_i is to take $r_0 \to \infty$ on one of the classical manifolds X_i. Our assumption could be wrong, for example, if there are additional smooth manifolds of G_2 holonomy that are likewise asymptotic to a cone on Y. In section 6, we will meet cases in which the enumeration of the possible singularities contains some surprises.

Now we can explain why there must be corrections to the classical limit described in (4.11). The curve described in (4.11) has an end with $\eta_{i-1} \to \infty$, $\eta_{i+1} \to 0$, and a second end with $\eta_{i-1} \to 0$, $\eta_{i+1} \to \infty$. The first end has the f_i diverging as a positive multiple of $(1, 1, -2)$ or a cyclic permutation thereof, but at the second end, the f_i diverge as a *negative* multiple of $(1, 1, -2)$. This does not correspond to any known classical limit of the theory, and according to our hypothesis, there is nowhere in the moduli space that the f_i diverge in this way. So (4.11) cannot be the exact answer.

On the other hand, a holomorphic function that has a pole also has a zero, so a component of $\mathcal{N}$ that contains a point with $\eta_{i-1} \to \infty$ must also contain a point with $\eta_{i-1} \to 0$. By our hypothesis, this must come from a classical limit associated with one of the X_i. By the Σ_3 symmetry, if two ends are in the same component, the third must be also, so given our assumptions, we have proved that $\mathcal{N}$ has a single component that contains all three ends. Therefore, it is possible to interpolate between X_1, X_2, X_3 without a phase transition.

On any additional branches of $\mathcal{N}$, the η_i have neither zeroes nor poles and hence are simply constant. This would correspond to a hypothetical quantum M-theory vacuum that is asymptotic to a cone on Y but whose "interior" has no classical limit, perhaps because it has a frozen singularity (analogous to frozen singularities that will appear in sections 6.3 and 6.4). If such a component exists, new tools are needed to understand it. We have no way to probe for the existence of such vacua in M-theory, and will focus our attention on the known branch of $\mathcal{N}$ that interpolates between the three classical manifolds X_i. In fact, we will henceforth use the name $\mathcal{N}$ to refer just to this branch.

We know of three such points P_i, $i = 1, 2, 3$ corresponding to the points at which one observes the classical manifolds X_i with large r_0. Near P_i, a local holomorphic parameter is expected to be the expansion parameter for membrane instantons on X_i. In fact, the three-sphere Q_i at the "center" of X_i (defined by setting $r = r_0$ in (2.23)) is a supersymmetric cycle. The amplitude for a membrane instanton wrapped on this cycle is

$$u = \exp\left(-TV(Q_i) + i\int_{Q_i} C\right). \tag{4.17}$$

Here T is the membrane tension, and $V(Q_i)$ is the volume of Q_i. For an antimembrane instanton, the phase $\int_{Q_i} C$ in (4.17) has the opposite sign. In any event, to define the sign of $\int_{Q_i} C$, one must be careful with the orientation of Q_i. According to (4.12), we know already that at P_i, η_{i-1} has a pole and η_{i+1} has a zero. It must then be that $\eta_{i-1} \sim u^{-s}$ and $\eta_{i+1} \sim u^t$ near P_i, with some $s, t > 0$. To determine s and t, we need only compare the phase of $\eta_{i\pm1}$, which is $\int_{D_{i\pm1}} C$, to the phase $\int_{Q_i} C$ of u. We saw in section 2.5 that Q_i is homologous (depending on its orientation) to $\pm D_{i-1}$ and to $\mp D_{i+1}$, so $s = t = 1$. Thus, η_{i-1} has a simple pole, and η_{i+1} has a simple zero. It should also be clear that the constant k in the definition of the η's could be determined (in terms of T) by comparing the modulus of $\eta_{i\pm1}$ to that of u.

Now we have enough information to describe $\mathcal{N}$ precisely. Each η_i has a simple pole at P_{i+1}, a simple zero at P_{i-1}, and no other zeroes or poles. The existence of a holomorphic function with just one zero and one pole implies that $\mathcal{N}$ is of genus zero. We could pick any i and identify $\mathcal{N}$ as the complex η_i plane (including the point at infinity), but proceeding in this way would obscure the Σ_3 symmetry. Instead, we pick an auxiliary parameter t such that the points P_i are at $t^3 = 1$, with the goal of expressing everything in terms of t. The action of Σ_3 on t can be determined from the fact that it must permute the cube roots of 1. Thus, Σ_3 is generated by an element of order three

$$t \to \omega t, \quad \omega = \exp(2\pi i/3), \tag{4.18}$$

and an element of order two,

$$t \to 1/t. \tag{4.19}$$

The antiholomorphic symmetry (4.14) will be

$$t \to 1/\bar{t}. \tag{4.20}$$

We identify P_i with the points $t = \omega^{i+1}$. η_i should equal 1 at P_i and should have a simple pole at P_{i+1} and a simple zero at P_{i-1}. This gives

$$\eta_i = -\omega \frac{t - \omega^i}{t - \omega^{i-1}}. \tag{4.21}$$

This is obviously invariant under the cyclic permutation of the η_i together with $t \to \omega^{-1} t$. It is invariant under elements of Σ_3 of order two since

$$\eta_1(1/t) = \eta_3(t)^{-1}, \;\; \eta_2(1/t) = \eta_2(t)^{-1}. \tag{4.22}$$

It is invariant under the antiholomorphic symmetry since

$$\eta_i(1/\bar{t}) = \overline{\eta}_i(t). \tag{4.23}$$

Finally, (4.21) implies the expected relation

$$\eta_1 \eta_2 \eta_3 = -1. \tag{4.24}$$

Thus, we have a unique candidate for $\mathcal{N}$, and it has all the expected properties.

Superpotential?

Another holomorphic quantity of interest is the superpotential W that arises from the sum over membrane instantons that are wrapped, or multiply wrapped, on the supersymmetric cycle $Q_i \subset X_i$. (For each $i = 1, 2, 3$, this method of computing W is valid near the point $P_i \in \mathcal{N}$ that describes the manifold X_i with large volume.) If the conical singularity that we have been studying is embedded in a compact manifold $\widehat{X}$ of G_2 holonomy, then the moduli of $\widehat{X}$, including the volume of Q_i, are dynamical, and the superpotential has a straightforward physical interpretation: it determines which points in the parameter space actually do correspond to supersymmetric vacua.

The physical interpretation of the superpotential is less compelling in the case considered in this paper of an asymptotically conical X, since the variables on which the superpotential depends are nondynamical, because of the infinite kinetic energy in their fluctuations, and behave as coupling constants in an effective four-dimensional theory rather than as dynamical fields. If there were more than one quantum vacuum for each point in $\mathcal{N}$, then the differences between the values of W for the different vacua would give tensions of BPS domain walls; but there is actually only one vacuum for each point in $\mathcal{N}$.

At any rate, let us see how far we can get toward determining W. W must have a simple zero at each of the P_i, since it vanishes in the absence of membrane instantons, and in an expansion in powers of instantons, it receives a one-instanton contribution proportional to the instanton coupling parameter u. (The analysis in [25] makes it clear that, since Q is an isolated and nondegenerate supersymmetric cycle, an instanton wrapped once on Q makes a nonvanishing contribution to the superpotential.) Since W has at least three zeroes, it has at least three poles. If we assume that the number of poles is precisely three, we can determine W uniquely. The positions of the three poles must form an orbit of the group Σ_3, and so these points must be permuted both by $t \to \omega t$ and by $t \to 1/t$. The only possibility (apart from $t = \omega^i$ where we have placed the zeroes of W) is that the poles are at $t = -\omega^i$, $i = 1, 2, 3$. The superpotential is then

$$W = ic\frac{t^3 - 1}{t^3 + 1}, \tag{4.25}$$

where the constant c could be determined from a one-instanton computation using the analysis in [25].

Note that $W(\omega t) = W(t)$, but

$$W(1/t) = -W(t), \tag{4.26}$$

so that the transformations of order two in Σ_3 are R-symmetries that reverse the sign of W. This is expected for geometrical reasons explained in section 2.4. Also, $W(1/\bar{t}) = \overline{W}(t)$ if c is real, so this candidate for W is compatible with the real structure of the problem.

Thus, we have a minimal candidate for W that is fairly natural, but we do not have enough information to be sure it is right; one could consider another function with more zeroes and poles, at the cost of introducing some unknown parameters.

One important fact is clearly that W must have some poles. What is their physical significance? This is not at all clear. A rough analogy showing the possible importance of the question is with the "flop" transition of Type II conformal field theory. In that problem, there is a complex moduli space $\widetilde{\mathcal{N}}$, analogous to $\mathcal{N}$ in the problem studied in the present paper, that interpolates between the two possible small resolutions of the conifold singularity. On $\widetilde{\mathcal{N}}$ there is a natural holomorphic function F, the "Yukawa coupling," that has a pole at a certain point of $\widetilde{\mathcal{N}}$. (In fact, F only has a straightforward interpretation as a four-dimensional Yukawa coupling if the conifold singularity is embedded in a compact Calabi-Yau manifold; in the noncompact case, the relevant modes are not square-integrable and are nondynamical. This is analogous to the status of the superpotential in our present problem.) At the pole, the Type II conformal field theory becomes singular. The singularity was mysterious for some time, but it was ultimately understood [3,4] that at this point one can make a phase transition to a different branch of vacua, corresponding to the deformation (rather than small resolution) of the conifold. The poles in W might similarly be related to novel phenomena.[10]

4.4. Membrane Instantons

On the classical manifold X_i, the three-cycle D_i is a boundary and hence, in a classical supersymmetric configuration, $\alpha_i = \int_{D_i} C$ vanishes. As we have seen in section 4.1, to get a smooth curve $\mathcal{N}$ interpolating between the different classical limits, we need to find a quantum correction to this statement.

[10] For example, a familiar mechanism [51] for generating a pole in a superpotential in four dimensions involves an $SU(2)$ gauge theory with a pair of doublets, so perhaps the theory near the poles has a description in such terms.

D_i is defined by setting the radial coordinate r in (2.23) equal to a large constant t, which should be taken to infinity, and also imposing a certain relation on the $SU(2)$ elements g_i. For $r \to \infty$, X_i becomes flat, with the curvature at $r = t$ vanishing as $1/t^2$. The volume of D_i grows as t^3, so to get a nonzero value of α_i, C must vanish as $1/t^3$. Perturbative corrections to the classical limit vanish faster than this. Consider Feynman diagram contributions to the expectation value of C at a point $P \in X_i$. We assume P is at $r = t$ and ask what happens as $t \to \infty$. If all vertices in the diagram are separated from P by a distance much less than t, we may get a contribution to $\langle C(P) \rangle$ that is proportional to some three-form built locally from the Riemann tensor R and its covariant derivatives. Any such three-form vanishes faster than $1/t^3$ for $t \to \infty$. (R itself is of order $1/t^2$, so its covariant derivative DR is of order $1/t^3$. But a three-form proportional to DR vanishes using the properties of the Riemann tensor. Other expressions such as RDR are of higher dimension and vanish faster than $1/t^3$.) Things are only worse if we consider Feynman diagrams in which some of the vertices are separated from P by a distance comparable to t. Such diagrams can give nonlocal contributions; those vanish at least as fast as t^{-9}, which is the order of vanishing at big distances of the massless propagator in eleven dimensions.

More fundamentally, the reason that perturbative corrections on X_i do not modify α_i is holomorphy. As we have seen, α_i is the argument of the holomorphic observable η_i, while a local holomorphic parameter at X_i is the membrane amplitude

$$u = \exp\left(-T\int_{Q_i} d^3x\sqrt{g} + i\int_{Q_i} C\right). \tag{4.27}$$

η_i must be a holomorphic function of u, and perturbative corrections to α_i or η_i must vanish as they are functions only of $|u|$, being independent of the argument $\int_{Q_i} C$ of u.

This makes it clear where, in M-theory on $\mathbf{R}^4 \times X_i$, we must look to find a correction to the statement that $\alpha_i = 0$. The correction must come from membrane instantons, that is from membranes whose world-volume is $y \times Q_i$, with y a point in $\mathbf{R}^4$. [11]

The quantity u is really a superspace interaction or superpotential. To convert it to an ordinary interaction, one must integrate over the collective coordinates of the membrane instanton. This integration is $\int d^4y\, d^2\theta$, where $d^2\theta$ is a chiral superspace integral over the fermionic collective coordinates of the membrane, and the y integral is the integral over the membrane position in $\mathbf{R}^4$.

[11] Using instantons to deform a moduli space is familiar in four-dimensional supersymmetric gauge theories [18,52,53].

Presently we will show that $\int d^4y d^2\theta\, u$ can be replaced by a (u-dependent) constant times $\int_{\mathbf{R}^4\times Q_i} *G$. Here, $*$ is the Hodge duality operator, so, in eleven dimensions, $*G$ is a seven-form that is integrated over the seven-manifold $\mathbf{R}^4 \times Q_i$. We postpone the evaluation of $\int d^4y\, d^2\theta\, u$ momentarily, and first concentrate on showing that, if the result is as claimed, this will solve our problem.

We will show that adding to the effective action a multiple of $\int_{\mathbf{R}^4\times Q_i} *G$ will induce a nonzero value for $\int_{D_i} C$. For this, we must analyze the correlation function

$$\left\langle \int_{\mathbf{R}^4\times Q_i} *G \cdot \int_{D_i} C \right\rangle, \tag{4.28}$$

and show that it is nonzero.

First of all, let us check the scaling. For propagation a large distance t, the two point function $\langle G \cdot C\rangle$ is proportional to $1/t^{10}$. But the integration in (4.28) is carried out over the seven-manifold $\mathbf{R}^4 \times Q_i$ times the three-manifold D_i, and so altogether over ten dimensions. Hence the powers of t cancel out, and also C can be treated as a free field, since corrections to the free propagator would vanish faster than $1/t^{10}$.

In the free field approximation, the action for C is a multiple of $\frac{1}{2}\int d^{11}x\sqrt{g}|G|^2$. The free field equations of motion, in the absence of sources, are $dG = d * G = 0$.

A simple way to evaluate the correlation function is to think of $\int_{D_i} C$ as a source that creates a classical G-field, after which $*G$ is then integrated over $\mathbf{R}^4 \times Q_i$. Thus, we look for the classical solution of the action

$$\frac{1}{2}\int_{\mathbf{R}^4\times X_{1,\Gamma}} |G|^2 + \int_{D_i} C. \tag{4.29}$$

The classical field created by the source is determined by the equations

$$\begin{aligned} dG &= 0 \\ d * G &= \delta_{D_i}. \end{aligned} \tag{4.30}$$

The first equation is just the Bianchi identity. The second contains as a source δ_{D_i}, a delta function form that is Poincaré dual to D_i.

We do not need to solve for G in detail. In order to evaluate $\int_{\mathbf{R}^4\times Q_i} *G$, it suffices to know $*G$ modulo an exact form. Any solution of the second equation in (4.30) (with G vanishing fast enough at infinity) will do. A convenient solution can be found as follows.

Let B be a ball in X_i whose boundary is D_i. (Existence of B is the reason for the classical relation $\int_{D_i} C = 0$!) We can obey $d * G = \delta_{D_i}$ by $*G = \delta_B$. Hence

$$\int_{\mathbf{R}^4 \times Q_i} *G = \int_{\mathbf{R}^4 \times Q_i} \delta_B. \tag{4.31}$$

The latter integral just counts the intersection number of the manifolds $\mathbf{R}^4 \times Q_i$ and B; it is the number of their intersection points, weighted by orientation. If B is obtained by "filling in" D_i in the obvious way, then there is precisely one intersection point, so the integral is 1.

The intersection number of B with $\mathbf{R}^4 \times Q_i$ is a version, adapted to this noncompact situation, of the "linking number" of the submanifolds $\mathbf{R}^4 \times Q_i$ and D_i. In essence, we have deduced the desired result about the deformation of the moduli space from this linking number.

Evaluation Of Superspace Integral

It remains to show that the integral $\int d^4y\, d^2\theta\, u$ has the right properties for the above computation. We write the two components of θ as θ^1 and θ^2, so $d^2\theta = d\theta^1\, d\theta^2$.

Let us write $u = e^w$, with $w = -TV(Q_i) + i\int_{Q_i} C$. Since a fermion integral has the properties of a derivative or a derivation, we have $\int d^2\theta\, u = u\left(\int d^2\theta w + 2\int d\theta^1 w \int d\theta^2 w\right)$. Here the second term $\int d\theta^1 w \int d\theta^2 w$ contributes a fermion bilinear (analogous to a Yukawa coupling in four-dimensional supersymmetric field theory). It can be omitted for our present purposes, since it can contribute to $\int_{D_i} C$ only via Feynman diagrams containing one boson and two fermion propagators, and such diagrams vanish for large t much faster than $1/t^{10}$.

We are left with computing $\int d^2\theta w = \int d^2\theta \left(-TV(Q_i) + i\int_{Q_i} C\right)$. Because w is a chiral or holomorphic field, we would have $\int d^2\theta\, \overline{w} = 0$, and hence

$$\int d^2\theta\, w = -2T\int d^2\theta V(Q_i) = 2i\int d^2\theta \int_{Q_i} C. \tag{4.32}$$

Because it is slightly shorter, we will compute $\int d^2\theta V(Q_i)$. But obviously, the result also determines $\int d^2\theta \int_{Q_i} C$. This fact will be useful in section 6.1, where we will need the latter integral.

Let ϵ be a covariantly constant spinor on $\mathbf{R}^4 \times X_i$. Under a supersymmetry generated by ϵ, the variation of the volume $V(Q_i) = \int_{Q_i} d^3x\sqrt{g}$ is, using the supersymmetry transformation laws of eleven-dimensional supergravity [54],

$$\delta_\epsilon V(Q_i) = -i\kappa \int_{Q_i} d^3x\sqrt{g} g^{ab} \overline{\epsilon}\Gamma_a\psi_b. \tag{4.33}$$

Here indices a, b, c run over tangent directions to Q_i, while indices A, B, C will run over tangent directions to $\mathbf{R}^4 \times X_i$. Also, κ is the gravitational coupling, ψ the gravitino, Γ_A are gamma matrices, and likewise $\Gamma^{A_1 A_2 \dots A_k}$ will denote an antisymmetrized product of gamma matrices.

To compute $\int d^2\theta\, V$, we let ϵ_1 and ϵ_2 be covariantly constant spinors of positive chirality on $\mathbf{R}^4$, and compute the second variation $\delta_{\epsilon_2}\delta_{\epsilon_1} V$. ($\epsilon_1$ and ϵ_2 are the tensor products of the same covariantly constant spinor on the G_2 manifold X_i times a constant positive chirality spinor on $\mathbf{R}^4$.) If ϵ_1 and ϵ_2 are properly normalized, this equals $\int d^2\theta\, V$. Ignoring terms proportional to ψ^2 (as their contributions vanish too fast for large t), we get

$$\delta_{\epsilon_2}\delta_{\epsilon_1} V(Q_i) = \frac{\kappa}{144}\int_{Q_i} d^3x\sqrt{g}\overline{\epsilon}_1\Gamma_a\left(\Gamma^{ABCDa} - 8\Gamma^{BCD}\delta^A_a\right)\epsilon_2\, G_{ABCD}. \tag{4.34}$$

The field G created by a delta function source on $y \times D_i$ has all indices tangent to X_i (or more precisely $y \times X_i \subset \mathbf{R}^4 \times X_i$) since the source has that property. Moreover, it follows from the symmetries of X_i that when restricted to Q_i, G can be written

$$G = G' + G'', \tag{4.35}$$

where G' has all four indices normal to D_i and G'' has precisely two indices in the normal directions. To make the notation simple in justifying this claim, take $i = 1$ so we are on X_1. Then D_1 is the set $(g_1, 1, 1)$, and is mapped to itself by $g_1 \to ug_1$, $u \in SU(2)$, so G has that symmetry. This symmetry acts trivially on Q_1, which (if we gauge away g_3 and then set g_1 to zero to get Q_1) is the set $(0, g_2, 1)$; the symmetry transforms the normal bundle to Q_1 in the fundamental representation of $SU(2)$ (which is of complex dimension two or real dimension four). As there are no odd order invariants in this representation, invariance of G under this $SU(2)$ action implies that all terms in G, when restricted to Q_1, have an even number of indices in the directions tangent to Q_1; the number can only be zero or two as Q_1 is three-dimensional.

Using this decomposition, we can simplify (4.34), getting

$$\delta_{\epsilon_2}\delta_{\epsilon_1}V(Q_i) = \frac{\kappa}{24}\int_{Q_i} d^3x\sqrt{g}\overline{\epsilon}_1\Gamma^{ABCD}\epsilon_2 G'_{ABCD} - \frac{\kappa}{48}\int_{Q_i} d^3x\sqrt{g}\overline{\epsilon}_1\Gamma^{ABCD}\epsilon_2 G_{ABCD}. \tag{4.36}$$

In fact, with ϵ_1 and ϵ_2 being covariantly constant spinors,

$$\overline{\epsilon}_1\Gamma^{ABCD}\epsilon_2 G_{ABCD} = 0. \tag{4.37}$$

This follows from the following facts. The space Ω^4 of four-forms on a G_2-manifold has a decomposition as $\Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}$, where the subscript refers to the transformation under the group G_2 acting in the tangent space at a point. This decomposition is described in [20], sections 3.5 and 10.4, where it is proved (Theorem 3.5.3) that it commutes with the Laplacian $\Delta = d^*d + dd^*$. The statement (4.37) means that G has no component in Ω^4_1, since with the ϵ's being covariantly constant spinors, $\Psi^{ABCD} = \overline{\epsilon}_1\Gamma^{ABCD}\epsilon_2$ is a covariantly constant antisymmetric tensor, and contracting it with G is the projection onto Ω^4_1. Since (upon solving (4.30)) the G-field produced by a source δ_{D_i} is

$$G = \frac{1}{\Delta} d * \delta_{D_i}, \tag{4.38}$$

to show that G has no component in Ω^4_1, it suffices to prove that $d * \delta_{D_i}$ has no component in Ω^4_1. It is equivalent to show that, if Υ is the covariantly constant three-form of the G_2-manifold, then

$$\Upsilon \wedge d * \delta_{D_i} = 0. \tag{4.39}$$

In fact, if we suitably normalize the ϵ's, then $\Upsilon = *\Psi$ (where we have "lowered indices" to interpret Ψ as a four-form) so contracting $d * \delta_{D_i}$ with Ψ is equivalent to taking a wedge product with Υ.

To verify (4.39), we again simplify the notation by choosing $i = 1$ and work on X_1. Using the description of Y by group elements a, b, c with $abc = 1$, D_1 is given by the equations $r = t$ and $a = 1 = bc$. δ_{D_1} is then a multiple of $\delta(r-t)\delta^3(a-1)\,dr\,\mathrm{Tr}(a^{-1}da)^3$, and $*\delta_{D_1}$ is a multiple of $\delta(r-t)\delta^3(a-1)\,\mathrm{Tr}\,(b^{-1}db)^3$. Finally, $d * \delta_{D_1}$ is a multiple of

$$d(\delta(r)\delta^3(a-1))\,\mathrm{Tr}\,(b^{-1}db)^3. \tag{4.40}$$

Now we use the explicit description of Υ in [21], eqn. (6.10), where it is called $Q_{(3)}$:

$$Q_{(3)} = e^0 \wedge e^i \wedge e^{\overline{i}} + \frac{1}{2}\epsilon_{ijk} e^i \wedge e^{\overline{j}} \wedge e^{\overline{k}} - e^1 \wedge e^2 \wedge e^3. \tag{4.41}$$

Here $e^i = \gamma\Sigma^i$ is a multiple of what in our notation is db. (For more details on the relation of our notation to that in [21], see the footnote after eqn. (2.23).) Since every term in $\Upsilon = Q_{(3)}$ is proportional to at least one factor of db, and $*\delta_{D_1}$ has three of them, which is the maximum possible, $\Upsilon \wedge \delta_{D_1} = 0$.

So we can reduce (4.36) to

$$\delta_{\epsilon_2}\delta_{\epsilon_1} V(Q_i) = \frac{\kappa}{24}\int_{Q_i} d^3x\sqrt{g}\Psi^{ABCD}G'_{ABCD}. \tag{4.42}$$

Now we have to use the fact that Q_i is a supersymmetric or calibrated cycle. This means that Υ is the volume form of Q_i. It also means with G' being a four-form in the normal directions, the map $G' \to *G'$, when restricted to Q', is equivalent to $G' \to \Upsilon \cdot \Psi^{ABCD}G_{ABCD}$. Here $*$ is understood as the Hodge duality operator in the seven-dimensional sense. So finally, (4.42) is a constant multiple of $\int_{Q_i} *G$.

Finally, when we incorporate the collective coordinate describing the membrane position $y \in \mathbf{R}^4$ and integrate over y, we get $\int_{\mathbf{R}^4\times Q_i} *G$, where now $*$ is understood in the eleven-dimensional sense.

5. Role Of A Fermion Anomaly

In any careful study of the C-field in M-theory, one encounters a fermion anomaly. A brief explanation of the reason is as follows. Let us call spacetime M. Let Q be a three-dimensional submanifold of M, and consider a membrane whose worldvolume is Q. In the worldvolume path integral for such a membrane, we meet a classical phase factor $\exp(i\int_Q C)$. But we also meet a fermion path integral. The classical phase factor must really be combined with a sign coming from the fermion path integral. It turns out that only their product is well-defined.

To describe the worldvolume fermions, let N_Q denote the normal bundle to Q in M. For simplicity, we will here assume that M and Q are spin, but in any event, for M-theory membranes, N_Q is always spin. This being so, we let $S(N_Q)$ be the spinor bundle of N_Q and decompose it in pieces of definite chirality as $S(N_Q) = S_+(N_Q) \oplus S_-(N_Q)$. Since N_Q has rank 8, $S_+(N_Q)$ is real and has rank eight. The worldvolume fermions are spinors on Q with values in $S_+(N_Q)$. We let $\mathcal{D}$ denote the Dirac operator on Q with values in $S_+(N_Q)$. The fermion path integral is the square root of the determinant of $\mathcal{D}$, or as we will write it, the Pfaffian of $\mathcal{D}$, or $\mathrm{Pf}(\mathcal{D})$. Because spinors on a three-manifold are pseudoreal, and

$S_+(N_Q)$ is real, $\mathrm{Pf}(\mathcal{D})$ is naturally real. Its absolute value can be naturally defined using zeta function regularization. But there is no natural way to define the sign of $\mathrm{Pf}(\mathcal{D})$. One cannot remove this indeterminacy by arbitrarily declaring $\mathrm{Pf}(\mathcal{D})$ to be, say, positive, because in general as Q moves in M, eigenvalue pairs of $\mathcal{D}$ can pass through zero and one wants $\mathrm{Pf}(\mathcal{D})$ to change sign. When Q is followed around a one-parameter family, $\mathrm{Pf}(\mathcal{D})$ may in general come back with the opposite sign. In that case, the fermion path integral has an anomaly which one cancels by modifying the quantization law for the periods of the curvature $G = dC$. The modified quantization law [55] says that for any four-cycle B in M,

$$\int_B \frac{G}{2\pi} = \frac{1}{2}\int_B \frac{p_1(M)}{2} \bmod \mathbf{Z}, \tag{5.1}$$

where here for a spin manifold M, $p_1(M)/2$ is integral but may not be even.

Mathematically, one can define a real line bundle, the "Pfaffian line bundle," in which $\mathrm{Pf}(\mathcal{D})$ takes values. Here we will focus on the fact that $\mathrm{Pf}(\mathcal{D})$ appears in the worldvolume path integral together with the classical phase factor coming from the C-field. It is really the product

$$\mathrm{Pf}(\mathcal{D}) \exp\left(i\int_Q C\right) \tag{5.2}$$

that must be well-defined. This means that $\exp\left(i\int_Q C\right)$ is not well-defined as a number; it must take values in the (complexified) Pfaffian line bundle. If we define $\mu(Q)$ to be 0 or 1 depending on whether $\mathrm{Pf}(\mathcal{D})$ is positive or negative, then the phase of the path integral is really

$$\phi(Q) = \int_Q C + \pi\mu(Q) \bmod 2\pi. \tag{5.3}$$

In general, only the sum of these two terms is well-defined.

This implies that the geometrical nature of the C-field is somewhat more subtle than one might have assumed; it is not the three-form analog of a $U(1)$ gauge field but of a Spin^c structure. One can make this analogy rather precise. For a spin 1/2 particle propagating around a loop $S \subset M$ and interacting with a "$U(1)$ gauge field" A, the phase of the path integral comes from a product

$$\mathrm{Pf}(D/Dt) \exp\left(i\int_S A\right), \tag{5.4}$$

which is the analog of (5.2). Here t is an angular parameter on S, and D/Dt is the Dirac operator on S acting on sections of the tangent bundle to M. A spin structure on M gives a way of defining the sign of $\mathrm{Pf}(D/Dt)$. On a spin manifold, A is an ordinary $U(1)$ gauge field and the two factors in (5.4) are separately well-defined. In the Spin^c case, there is no definition of the sign of $\mathrm{Pf}(D/Dt)$ as a number, the geometrical meaning of A is modified, and only the product in (5.4) is well-defined.

Because only the total phase $\phi(Q)$ is well-defined, the definition of the periods $\alpha_i = \int_{D_i} C$ in section 4 should be modified to

$$\alpha_i = \int_{D_i} C + \pi\mu(D_i). \tag{5.5}$$

The goal of the present discussion is to determine how the correction to the definition of α_i should enter the formulas in section 4.

For our present purposes, we do not need to know much about Pfaffian line bundles, because everything we need can be deduced from a situation in which the separate contributions to the phase actually *are* well-defined. This is the case in which we are given a four-manifold B in spacetime with boundary Q. For simplicity we will assume B to be spin. Then we would like to write

$$\exp\left(i\int_Q C\right) = \exp\left(i\int_B G\right). \tag{5.6}$$

The right hand side is well-defined, as G is gauge-invariant. However, the right hand side may depend on the choice of B; according to (5.1), this will occur when $p_1(M)/2$ is not even. At any rate, since a choice of B enables us to make a natural definition of $\int_Q C$ mod 2π, and since the total phase $\phi(Q)$ is always well-defined, it must be that once B is chosen, one also has a natural definition of μ. The appropriate definition (which was originally pointed out by D. Freed) is as follows. Let N_B be the normal bundle to B in M. Let $S(N_B)$ be the spin bundle of N_B; it is a real bundle of rank eight that on Q reduces to $S_+(N_Q)$. Let $\mathcal{D}_B$ be the Dirac operator on B with values in $S(N_B)$, and with Atiyah-Patodi-Singer (APS) boundary conditions along Q. Its index is even, since in general, the Dirac index in four dimensions with values in a real bundle, such as $S(N_B)$, is even. Let $i(B)$ be the index of $\mathcal{D}_B$ and let $\nu(B) = i(B)/2$. Then in this situation, we define

$$\mu(Q) = \nu(B) \bmod 2. \tag{5.7}$$

The justification for this definition is that if (5.6) and (5.7) are used, one gets a definition of the total phase $\phi(Q)$ that is independent of the choice of B.

This is proved as follows. If B_1 and B_2 are two spin manifolds in M with boundary Q, one forms the closed four-manifold $B = B_1 - B_2$, where the minus sign refers to a reversal of orientation of B_2 so that B_1 and B_2 join smoothly on their common boundary. The gluing theorem for the APS index gives $\nu(B_1) - \nu(B_2) = \nu(B)$. The index theorem for the Dirac operator on a closed four-manifold gives $\nu(B) = \frac{1}{2}\int_B (p_1(M)/2)$ mod 2, and then using (5.1), this implies that when B_1 is replaced by B_2, the change in the period $\int_Q C$ just cancels the change in $\mu(Q)$. We shall apply a variant of this argument later, in section 5.2, to give an explicit topological formula for $\nu(B)$.

Before presenting some relevant examples in which there is an anomaly, let us describe some simple cases in which an anomaly involving the index $\nu(B)$ does *not* appear. The most basic case is that Q is a copy of $\mathbf{S}^3$, embedded in $M = \mathbf{R}^{11}$ in the standard way. Then we can take B to be a four-ball, with a standard embedding in $\mathbf{R}^4$. In this case, N_B is a rank seven bundle with a trivial flat connection, and $S(N_B)$ is a trivial flat bundle of rank 8. So $i(B)$ is divisible by 8, and hence $\nu(B)$ is zero mod 2. (In fact, it can be shown that in this example, $i(B)$ vanishes.)

This example has the following generalization. Let $M = \mathbf{R}^4 \times X$, with any seven-manifold X. Suppose that Q and B are submanifolds of X. Then N_B is a direct sum $\mathbf{R}^4 \oplus N'$, where N' is the rank three normal bundle to B in X, and $\mathbf{R}^4$ is a trivial flat bundle of rank four. In this situation, $S(N_B)$ is (when complexified) the sum of four copies of $S(N')$ (the spinors of N'), so $i(B)$ is divisible by four and hence $\nu(B)$ is even and $\mu(Q) = 0$.

Let now X_i be one of the three familiar seven-manifolds of G_2 holonomy that is asymptotic to a cone on $\mathbf{S}^3 \times \mathbf{S}^3$. Let D_i be a three-sphere in $\mathbf{S}^3 \times \mathbf{S}^3$ that bounds a ball B in X_i. Classically, the curvature $G = dC$ vanishes for supersymmetry, and hence $\int_{D_i} C = \int_B G = 0$. Moreover, from what has just been observed, $\nu(B)$ is zero mod 2 in this example, and $\mu(D_i) = 0$. So finally we learn that the "period" α_i, correctly defined as in (5.5), vanishes. This completes the justification of the assertion made in section 4 that $\alpha_i = 0$ on X_i in the classical limit.

Now let us consider a somewhat analogous question that arose in section 4. In $\mathbf{S}^3 \times \mathbf{S}^3$, we defined three-spheres D_1, D_2, D_3, with $D_1 + D_2 + D_3 = 0$ in $H_3(\mathbf{S}^3 \times \mathbf{S}^3; \mathbf{Z})$. It follows that

$$\sum_i \int_{D_i} C = 0, \tag{5.8}$$

since the left hand side can be written as $\int_B G$, where B is a four-dimensional chain in $\mathbf{S}^3 \times \mathbf{S}^3$ with boundary $D_1 + D_2 + D_3$.[12] We have therefore

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi \sum_i \mu(D_i). \tag{5.9}$$

We claim that $\sum_i \mu(D_i) = 1$, and hence that

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi. \tag{5.10}$$

To demonstrate this, we first need to describe some properties of the index function $\nu(B)$ and some methods for calculating it.

5.1. Stiefel-Whitney Classes

The mod 2 invariants, such as $\nu(B)$, that we shall be dealing with are best described in terms of the Stiefel-Whitney classes w_i. These are, in a sense, the real counterparts of the more familiar Chern classes c_i. They may be less familiar, and so we shall briefly review them at this stage.

For an $O(n)$ bundle (or equivalently for a real vector bundle) over a space Y, the w_i are characteristic classes

$$w_i \in H^2(Y; \mathbf{Z}_2) \tag{5.11}$$

such that $w_0 = 1,\ w_i = 0$ for $i > n$. They have the following properties:

(1) w_1 measures the obstruction to orientability of a vector bundle; to a circle $C \subset Y$ it assigns the value 1 or -1 depending on whether the restriction of the bundle to C is orientable. In particular, for the two-sheeted cover $\mathbf{S}^n$ of $\mathbf{RP}^n$, w_1 assigns the nontrivial element of $H^1(\mathbf{RP}^n; \mathbf{Z}_2)$.

(2) Let $w(E) = w_0(E) + w_1(E) + \ldots$ be the total Stiefel-Whitney class. For a direct sum of real vector bundles E, F we have the product formula

$$w(E \oplus F) = w(E) \cdot w(F). \tag{5.12}$$

[12] If one could pick B to be a smooth manifold in $\mathbf{S}^3 \times \mathbf{S}^3$, then $\sum_i \mu(D_i)$ would vanish by the argument given above. However, the D_i intersect, so B cannot be a manifold. Later, we will see that after perturbing the D_i slightly, we can take B to be a smooth manifold in $\mathbf{R}^2 \times \mathbf{S}^3 \times \mathbf{S}^3$, but this does not lead to vanishing of $\sum_i \mu(D_i)$.

Taking F to be a trivial bundle, $w(F) = 1$, and so the product formula implies that the w_i are *stable*, i.e. unchanged by $E \to E \oplus F$.

(3) For an $SO(2n)$-bundle, w_{2n} is the mod 2 reduction of the Euler class $e \in H^{2n}(Y; \mathbf{Z})$:

$$e \equiv w_{2n} \bmod 2. \tag{5.13}$$

Recall that, for Y a manifold, e may be defined by the locus of zeros of a generic section s of a rank $2n$ vector bundle. The fact that w_{2n} is the mod 2 reduction of e actually follows from the corresponding statement mod 2 for a generic section of any real vector bundle (oriented or not, even or odd-dimensional). More generally we have the following: For a real vector bundle E of rank n, let $s_1, ..., s_{n-i+1}$ be generic sections. Then $w_i(E)$ is represented by the mod 2 cycle of points where the sections become linearly dependent.

Note that the Chern classes may be defined by a similar property for sections of a complex vector bundle, except that dimensions are doubled and we work with integer cohomology.

The first two classes w_1, w_2 characterize orientability and spin, i.e. they vanish for $SO(n)$ and $\mathrm{Spin}(n)$ bundles, respectively. For a $\mathrm{Spin}(n)$-bundle we also have

$$w_3 = 0, \; w_4 = p_1/2. \tag{5.14}$$

Here p_1 is the first Pontrjagin class (which is naturally divisible by two for a spin-bundle). By the stability of the w_i (and of p_1), we can check (5.14) by looking at $\mathrm{Spin}(3)$ and $\mathrm{Spin}(4)$ bundles. But

$$\mathrm{Spin}(3) = SU(2), \quad \mathrm{Spin}(4) = SU(2) \times SU(2). \tag{5.15}$$

Since $\pi_i(SU(2)) = 0$ for $i \leq 2$, the first statement implies that a $\mathrm{Spin}(3)$-bundle over Y is always trivial on the three-skeleton of Y, showing that $w_3 = 0$. The second statement implies that a $\mathrm{Spin}(4)$-bundle has, in dimension 4, two integral characteristic classes (say a, b) coming from the Chern classes of the two factors. They are related to the Pontrjagin class p_1 and the Euler class e by

$$\begin{aligned} a &= \frac{p_1}{4} + \frac{e}{2} \\ b &= \frac{p_1}{4} - \frac{e}{2}. \end{aligned} \tag{5.16}$$

These formulae show that, as asserted in (5.14), $p_1/2$ is naturally an integral class, namely $2a - e$. Moreover, reducing modulo 2, and using the fact that w_4 is the mod 2 reduction of e, we deduce

$$p_1/2 = w_4 \quad \text{mod } 2. \tag{5.17}$$

Although we have only verified this for Spin(4)-bundles, it follows for all Spin(n)-bundles ($n > 4$). This is an aspect of stability: since $\mathrm{Spin}(n+1)/\mathrm{Spin}(n) = \mathbf{S}^n$, no new relations can be introduced on q-dimensional characteristic classes for $q \leq n$, when passing from n to $n+1$.

For our applications, (5.17) is the key formula, and the reason for our interest in Stiefel-Whitney classes.

For a Spin(3)-bundle, we have $w_4 = 0$. (This is a special case of the vanishing of w_k for a bundle of rank less than k.) This implies that, if a Spin(n)-bundle over Y can be reduced over a subspace Y_0 to a Spin(3)-bundle, then w_4 can be lifted back from $H^4(Y; \mathbf{Z}_2)$ to a *relative class* in $H^4(Y, Y_0; \mathbf{Z}_2)$. More precisely, a choice of reduction over Y_0 gives a definite choice of relative class. This is because such a reduction is given by $(n-3)$ sections $s_1, ..., s_{n-3}$ which are independent over Y_0: their locus of dependence then gives a representative cycle in $Y - Y_0$ for the relative w_4.

Actually we can define a relative w_4 in the more general situation of a Spin(n) bundle over Y with a reduction to an H-bundle over Y_0, where

$$H = \prod_i \mathrm{Spin}(n_i) \quad \sum_i n_i = n, \; n_i \leq 3. \tag{5.18}$$

(Such an H is not always a subgroup of Spin(n). It may be a finite covering of such, so a reduction means a reduction and a lifting.) Indeed, each Spin(n_i) bundle, with $n_i \leq 3$, has trivial Stiefel-Whitney classes ($w = 1$), and so by the product formula (5.12), $w = 1$ for an H-bundle. Thus again w_4 lifts back to a relative class. The uniqueness can be seen from the universal case when Y, Y_0 are the appropriate Grassmannians, using the fact that the Grassmannian for H (i.e. the product of the Grassmannians for the factors) has no cohomology in dimension 3.

Similar reasoning enables us to use the product formula in the relative case to show that if, E, F are spin-bundles over Y with reductions to groups of type H over Y_0, then the relative w_4 is additive:

$$w_4(E \oplus F) = w_4(E) + w_4(F). \tag{5.19}$$

Here one must use the fact that, for spin-bundles, $w_1 = w_2 = w_3 = 0$.

5.2. Topological formula for $\nu(B)$

After this digression about Stiefel-Whitney classes, we return to our problem of computing the index function $\nu(B)$ and the corresponding invariant $\mu(Q)$ introduced in eqn. (5.7).

We shall give a topological way of computing these mod 2 invariants under the assumption that at least near Q, our space-time manifold is $M = \mathbf{R}^5 \times Y$ where Y is a spin six-manifold and $Q \subset Y$. However we shall *not* assume that Q is the boundary of some $B \subset Y$ since, as we argued earlier, this would make our invariant automatically zero.

In fact, for our applications, we will allow Q to be not quite a smooth submanifold of Y, but the union of a number of smooth submanifolds

$$Q = Q_1 \cup Q_2 \cup ... \cup Q_k. \tag{5.20}$$

It may be that the Q_i intersect in Y, but if this happens, we can separate these components, so that they do not intersect, by using some of the additional $\mathbf{R}^5$-variables. Thus we take k distinct vectors $u_1, ..., u_k$ in $\mathbf{R}^5$ and shift the component Q_j to lie over the point u_j. With this understanding, Q becomes a genuine submanifold of M.

We now assume that B is a compact spin four-manifold, embedded in M, with boundary Q. This implies that, in homology,

$$Q = \sum_j Q_j = 0 \qquad \text{in } H_3(M) = H_3(Y). \tag{5.21}$$

We shall also assume that, near each Q_j, B is the product of Q_j with the half-line ru_j, $r \geq 0$, so that u_j is normal to Q_j in B.

Consider now the normal bundle N_B to B in M. This is a Spin(7)-bundle. Over each Q_j it splits off a trivial $\mathbf{R}^4$ factor (orthogonal to u_j in $\mathbf{R}^5$) and hence reduces to a Spin(3)-bundle. In this situation, as explained above, we have a relative class

$$w_4(N_B) \in H^4(B, Q; \mathbf{Z}_2). \tag{5.22}$$

We claim that

$$\nu(B) = w_4(N_B) \tag{5.23}$$

where on the right side we evaluate the relative class w_4 on the top cycle of B, to get an element of $\mathbf{Z}_2$.

To argue this, note first that over each component Q_j we have a natural decomposition

$$N_B|_{Q_j} = N_j \oplus R^4, \tag{5.24}$$

where N_j is the normal Spin(3)-bundle to Q_j in Y. Since $\dim Q_j = 3$, N_j is actually trivial, so that N_B also gets trivialized over $Q = \partial B$. Now take two copies of B, and put a rank 7 vector bundle over each copy. Over the first copy B_1 we take $N_1 = N_B$ while over the second copy B_2 we take the trivial bundle N_2. Gluing B_1 and $-B_2$ together to form a closed spin four-manifold $\widehat{B}$, we can also glue together the two vector bundles, using the trivialization coming from (5.24), to get a vector bundle $\widehat{N}$. We now compute the index of the Dirac operator, with coefficients in the spinors of this rank 7 bundle, for the two B_i and for $\widehat{B}$. We denote the spinors of N by $S(N)$, and for the manifolds with boundary B_i, we take APS boundary conditions. The additivity of the APS index gives

$$\mathrm{index} D_{B_1}(S(N_1)) - \mathrm{index} D_{B_2}(S(N_2)) = \mathrm{index} D_{\widehat{B}} S(\widehat{N}). \tag{5.25}$$

Now over B_2, N_2 is trivial, so the index is divisible by 8. On the closed manifold $\widehat{B}$, the index theorem gives

$$\frac{1}{2} \mathrm{index} D_{\widehat{B}}(S(\widehat{N})) = \frac{p_1(\widehat{N})}{2} \bmod 2, \tag{5.26}$$

where the right-side is viewed as an integer by evaluation on the top cycle of $\widehat{B}$.

Hence (5.25) gives

$$\nu(B) = \frac{p_1(\widehat{N})}{2} \quad \bmod 2. \tag{5.27}$$

But $p_1(\widehat{N})$ is just the relative Pontrjagin class of N_B given by the trivialization (5.24) on ∂B. As explained in section 5.1, this can be re-written in terms of the relative w_4 to give (5.23).

At this stage we could just as well have stuck to (half) the relative Pontrjagin class. The advantages of using w_4 will appear later.

5.3. Topological Formula For $\mu(Q)$

If we can explicitly find a convenient four-manifold $B \subset M$ with boundary Q, then (5.23) gives an effective way to calculate $\nu(B)$ and hence our anomaly $\mu(Q)$. We shall exhibit a concrete example of this later for the case when $Y = \mathbf{S}^3 \times \mathbf{S}^3$ and each Q_j is also

$\mathbf{S}^3$. However, as we shall now explain, it is possible to give a useful formula for $\mu(Q)$ for some cases even when we do not know how to construct B.

The first step is to observe that

$$N_B \oplus T_B = T_M|_B \tag{5.28}$$

where T_B, T_M are the respective tangent bundles. Moreover, we are in the general situation, explained in the discussion on Stiefel-Whitney classes, where we can apply the formula (5.19) for relative w_4 (here $H = \mathrm{Spin}(3)\times \mathrm{Spin}(3) \to \mathrm{Spin}(11)$), so that

$$w_4(N_B) + w_4(T_B) = \langle w_4(T_M), B\rangle. \tag{5.29}$$

Thus, in (5.23), we can replace $w_4(N_B)$ by the other two terms in (5.29). We examine each in turn.

The easy one is $w_4(T_B)$. By (5.13), we have

$$\begin{aligned} w_4(T_B) =& e(B, \partial B) \bmod 2 \\ =& \chi(B) \bmod 2, \end{aligned} \tag{5.30}$$

where χ is the usual Euler-characteristic. For a four-dimensional spin-manifold B with boundary, $\chi(B)$ mod 2 depends only on ∂B. This follows from the fact that χ is additive under gluing, while for a closed spin manifold χ is congruent to the signature mod 2, and hence even.[13] Notice that, in this elementary argument, given Q we can choose *any* spin manifold B with $\partial B = Q$. We do not need B to be embedded in M. In particular, if Q is a union of components Q_j, we can take B to be a disjoint union of B_j with $\partial B_j = Q_j$. Thus, if (as in our example) Q is a union of three three-spheres Q_j, we can take each B_j to be a four-ball so that $\chi(B_j) = 1$ and hence $\chi(B) \equiv 1 \bmod 2$.

It remains for us to dispose of the term in (5.29) coming from the relative w_4 of T_M. We would like to find conditions that make this zero. One can actually show that the absolute w_4 is always zero for $M = Y \times \mathbf{R}^5$, but we need the relative version. From the definition and properties of the relative w_4, we see that to show that this object vanishes, it is sufficient to find a reduction of T_M to Spin(3)$\times$ Spin(3) which agrees with the natural decomposition over each Q_j (with the first factor being tangent to Q_j and the second

[13] The signature of a four-dimensional spin-manifold is even (and in fact divisible by 16) by the index theorem.

normal to Q_j in Y). Essentially, all we need is a (spin) 3-dimensional sub-bundle of T_Y which is transversal to each Q_j. This is easy to do if we make the following further assumptions (which hold in the examples we need):

(A) $Y = Y_1 \times Y_2$, where the Y_i are three-dimensional spin-manifolds.

(B) Each Q_j is either a cross-section or a fibre of the projection $\pi_2 : Y \to Y_2$.

To get our sub-bundle of T_Y we start with the bundle $\pi_1^* T_{Y_1}$. This is certainly transversal to all cross-sections of π_2, but of course it is not transversal to the fibres of π_2. However spin three-manifolds are always parallelizable so that $\pi_1^* T_{Y_1} \cong \pi_2^* T_{Y_2}$. Choosing such an isomorphism we can rotate $\pi_1^* T_{Y_1}$ slightly in the Y_2-direction. If this rotation is small enough it does not destroy transversality to a finite set of cross-sections. But any nonzero rotation (with all "rotation angles" nonzero) gives us transversality to all fibres. Thus conditions *(A)* and *(B)* are sufficient to ensure that

$$\langle w_4(T_M), B \rangle = 0 \tag{5.31}$$

and hence (5.23),(5.29), and (5.30) give us our final formula

$$\mu(Q) = \chi(B) = \sum_j \chi(B_j) \bmod 2, \tag{5.32}$$

where B_j are any spin four-manifolds (not necessarily in M) with $\partial B_j = Q_j$.

5.4. The Examples

Our first example is the familiar case

$$Y = SU(2)^3 / SU(2). \tag{5.33}$$

There are three projections $\pi_j : Y \to \mathbf{S}^3$ (given by omitting the j^{th} coordinate), and the $\mathbf{S}^3$-fibres of the π_j we have denoted by D_j. We noted in section 2.5 that, in $H_3(Y; \mathbf{Z})$, we have

$$D_1 + D_2 + D_3 = 0. \tag{5.34}$$

We want to compute $\mu(Q)$ with $Q = D_1 \cup D_2 \cup D_3$.

D_2 and D_3 are both cross-sections of the projection π_1: in fact, if we identify Y with the product of the first two factors, then D_2 is the graph of a constant map $\mathbf{S}^3 \to \mathbf{S}^3$, while D_3 is the graph of the identity map.

Thus, the conditions *(A)* and *(B)* are satisfied, so that formula (5.32) applies. Taking B to be a union of balls bounded by the D_i, we get

$$\mu(Q) = 1 \bmod 2. \tag{5.35}$$

This establishes (5.10) as promised.

The second example is the generalization discussed in Section 2.5 and further explored in section 6. It involves the six-manifold

$$Y_\Gamma = \mathbf{S}^3/\Gamma \times \mathbf{S}^3, \tag{5.36}$$

where Γ is a finite subgroup of $SU(2)$. This can also be viewed as the quotient of

$$Y = SU(2)^3/SU(2) \tag{5.37}$$

by the action of Γ on the left on the first factor. The three projections of Y to $\mathbf{S}^3$ now give rise to projections

$$\begin{aligned} \pi_1' :& Y_\Gamma \to \mathbf{S}^3 \\ \pi_2' :& Y_\Gamma \to \mathbf{S}^3/\Gamma \\ \pi_3' :& Y_\Gamma \to \mathbf{S}^3/\Gamma. \end{aligned} \tag{5.38}$$

The fibres of these projections are denoted by $D_j' : D_1' = \mathbf{S}^3/\Gamma,\ D_2' = D_3' = \mathbf{S}^3$. Again D_2' and D_3' are cross-sections of the projection π_1'. The homology relation in $H_3(Y_\Gamma;\mathbf{Z})$ is given as in section 2.5 by

$$ND_1' + D_2' + D_3' = 0 \tag{5.39}$$

where N is the order of Γ.

We therefore take our three-manifold Q to have $N+2$ components

$$Q = Q_1 \cup Q_2 \cup \ldots \cup Q_{N+2}, \tag{5.40}$$

where the first N are parallel copies of D_1' (the fibre of π_1') and the last two are D_2' and D_3'. The conditions *(A)* and *(B)* are again satisfied, so that formula (5.32) gives

$$\mu(Q) = \sum_{j=1}^{N+2} \chi(B_j), \tag{5.41}$$

where B_j is any spin-manifold with boundary Q_j. For $j = N+1$ or $N+2$, we can take B_j to be a four-ball, while for $j \leq N$ we can take B_j to be the resolution of the singular complex surface $\mathbf{C}^2/\Gamma$. This has non-zero Betti numbers

$$\begin{aligned} b_0 &= 1 \\ b_2 &= r, \end{aligned} \tag{5.42}$$

where r is the rank of the corresponding Lie group (of type A, D or E). Thus, mod 2,

$$\mu(Q) = N(1+r). \tag{5.43}$$

But Nr is always even (in fact, N is even except for a group of type A with even r), so finally

$$\mu(Q) = N \bmod 2, \tag{5.44}$$

generalizing (5.35). This result will be used in section 6.2 (formula (6.7)).

Finally, we return to the first example of $Y = \mathbf{S}^3 \times \mathbf{S}^3$ and exhibit an explicit four-manifold B with boundary Q, with the aim of giving a more direct proof of (5.35). For this purpose we shall introduce the quaternionic projective plane $\mathbf{HP}^2$. By definition, this is the eight-manifold parameterized by triples of quaternions (u_1, u_2, u_3), not all zero, modulo right multiplication by an element of the group $\mathbf{H}^*$ of non-zero quaternions. This group is

$$\mathbf{H}^* = SU(2) \times \mathbf{R}^+, \tag{5.45}$$

the product of the unit quaternions and a radial coordinate. The subspace $\mathbf{HP}^2_0$ of $\mathbf{HP}^2$ in which the homogeneous quaternionic coordinates (u_1, u_2, u_3) are each nonzero is a copy of $\mathbf{H}^* \times \mathbf{H}^* = SU(2) \times SU(2) \times \mathbf{R}^2$. So our spacetime $M = \mathbf{R}^5 \times Y$ can be identified as $M = \mathbf{R}^3 \times \mathbf{HP}^2_0$. Y is naturally embedded in $\mathbf{HP}^2_0$ as the subspace whose homogeneous quaternionic coordinates (u_1, u_2, u_3) satisfy

$$|u_1| = |u_2| = |u_3|\,. \tag{5.46}$$

Similarly the three seven-manifolds X_i are also embedded in $\mathbf{HP}^2$. For instance, X_1 is given by

$$|u_1| < |u_2| = |u_3|\,. \tag{5.47}$$

The homology class D_j is represented in Y by the three-sphere with $u_k = 1$ for $k \neq j$. In $\mathbf{HP}^2_0$, D_j can be deformed to a three-sphere that links around the line $u_j = 0$ in $\mathbf{HP}^2$.

We get a simple choice for our required four-manifold B as follows. Let $\overline{B}$ be a quaternionic line in $\mathbf{HP}^2$ given by, say, the equation

$$u_1 + u_2 + u_3 = 0. \tag{5.48}$$

Now define B by removing small neighbourhoods of the three points where $\overline{B}$ meets the coordinate lines. Thus B is a four-sphere with three open balls removed.

It is now easy to compute the topological invariant $w_4(N_B)$, which by (5.23) determines our invariant $\nu(B)$. First we can replace the rank 7 bundle N_B by the rank 4 bundle N'_B which is just the normal to B in $\mathbf{HP}^2$ (since the remaining $\mathbf{R}^3$ of $M = Y \times \mathbf{R}^2 \times \mathbf{R}^3$ gives a trivial factor). The reduction of N'_B to a Spin(3)-bundle over $Q = \partial B$ amounts to fixing a normal direction to the compactification $\overline{B}$ at the three ends. The relative $w_4(N'_B)$ is thus just $w_4(N'_{\overline{B}})$, which is the reduction mod 2 of the Euler class of the normal bundle to $\mathbf{HP}^1$ in $\mathbf{HP}^2$. But this is just 1, since two quaternionic lines in the quaternionic plane meet in just one point. This shows that

$$\nu(B) = 1, \tag{5.49}$$

in agreement with (5.35).

5.5. More on the Quaternionic Projective Plane

As we have just seen, the quaternionic projective plane $\mathbf{HP}^2$ provides a convenient compact eight-manifold which naturally contains, as in (5.47), the three seven-manifolds X_i, and it enabled us to give a direct computation of the fermionic anomaly in one of the main examples. We shall now point out some further geometric properties of $\mathbf{HP}^2$, which are closely related to the discussion in section 3. (The rest of the paper does not depend on the following discussion.)

Since $\mathbf{HP}^2$ is the quotient of $\mathbf{H}^3 - \{0\}$ by the right action of $\mathbf{H}^*$, it admits the left action of $Sp(3)$ and, in particular, of its subgroup $U(3)$. The complex scalars $U(1) \subset U(3)$ therefore act on $\mathbf{HP}^2$ and commute with the action of $SU(3)$. We shall show that

$$\mathbf{HP}^2/U(1) = \mathbf{S}^7. \tag{5.50}$$

Note that the fixed-point set of $U(1)$ is $\mathbf{CP}^2$ (the subset of $\mathbf{HP}^2$ in which the homogeneous coordinates are all complex numbers), which has codimension 4. $U(1)$ acts on the normal bundle in the usual way, so that the quotient is indeed a manifold.